

ON M -SECTORIAL EXTENSIONS OF SECTORIAL OPERATORS

YU. M. ARLINSKIĬ AND A. B. POPOV

ABSTRACT. In our article [15] description in terms of abstract boundary conditions of all m -accretive extensions and their resolvents of a closed densely defined sectorial operator S have been obtained. In particular, if $\{\mathcal{H}, \Gamma\}$ is a boundary pair of S , then there is a bijective correspondence between all m -accretive extensions \tilde{S} of S and all pairs $\langle \mathbf{Z}, X \rangle$, where \mathbf{Z} is a m -accretive linear relation in \mathcal{H} and $X : \text{dom}(\mathbf{Z}) \rightarrow \text{ran}(\tilde{S}_F)$ is a linear operator such that:

$$\|Xe\|^2 \leq \text{Re}(\mathbf{Z}(e), e)_{\mathcal{H}} \quad \forall e \in \text{dom}(\mathbf{Z}).$$

As is well known the operator S admits at least one m -sectorial extension, the Friedrichs extension. In this paper, assuming that S has non-unique m -sectorial extension, we established additional conditions on a pair $\langle \mathbf{Z}, X \rangle$ guaranteeing that corresponding \tilde{S} is m -sectorial extension of S . As an application, all m -sectorial extensions of a nonnegative symmetric operator in a planar model of two point interactions are described.

CONTENTS

Introduction	2
1. Preliminaries	4
1.1. Sectorial forms and operators	4
1.2. The Friedrichs and Kreĭn-von Neumann m -sectorial extensions	5
1.3. Boundary triplets and abstract boundary conditions for quasi-selfadjoint extensions of nonnegative symmetric operator	7
2. Abstract boundary conditions for m -accretive extensions of sectorial operators	9
3. m -sectorial extensions	14
4. Nonnegative symmetric operator and its quasi-selfadjoint m -accretive extensions	20
5. m -sectorial extensions of a symmetric operator in the model of two point interactions on a plane	25
References	34

2010 *Mathematics Subject Classification.* 47A06, 47A07, 47A20, 47B25, 47B44, 82B23.

Key words and phrases. Sectorial operator, accretive operator, Friedrichs extension, Kreĭn -von Neumann extension.

INTRODUCTION

Let \mathfrak{H} be a complex Hilbert space with the inner product (\cdot, \cdot) . We use the symbols $\text{dom}(T)$, $\text{ran}(T)$, $\ker(T)$ for the domain, the range, and the null-subspace of a linear operator T . The resolvent set of a linear operator T is denoted by $\rho(T)$. The linear space of bounded operators acting between Hilbert spaces \mathfrak{H}_1 and \mathfrak{H}_2 is denoted by $\mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and the Banach algebra $\mathbf{L}(\mathfrak{H}, \mathfrak{H})$ by $\mathbf{L}(\mathfrak{H})$. A linear operator T in a complex Hilbert space \mathfrak{H} is called *accretive* if its numerical range

$$W(T) \stackrel{\text{def}}{=} \{(Tu, u), u \in \text{dom}(T), \|u\| = 1\}$$

is contained in the closed right half-plane, i.e.,

$$\text{Re}(Tu, u) \geq 0 \text{ for all } u \in \text{dom}(T).$$

An accretive operator T is called *maximal accretive* or *m-accretive*, if one of the following equivalent conditions holds [28, 37, 38]:

- 1) T is closed and has no accretive extensions in \mathfrak{H} ;
- 2) resolvent set $\rho(T)$ contains a point from an open left half-plane;
- 3) T is a closed densely defined operator and its adjoint T^* is an accretive operator;
- 4) the operator $-T$ generates one-parameter contractive semigroup $U(t) = \exp(-tT)$, $t \geq 0$.

One can prove the following equality for an arbitrary m -accretive operator T :

$$\ker(T) = \ker(T^*). \quad (0.1)$$

Let $\alpha \in [0, \pi/2)$. Denote by the $\Theta(\alpha)$ the sector in the complex plane

$$\Theta(\alpha) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |\arg z| \leq \alpha\}.$$

A linear operator S is called *sectorial* with the vertex at the origin and the semi-angle α [28] if $W(S) \subseteq \Theta(\alpha)$. Clearly, S is sectorial if and only if:

$$|\text{Im}(Sx, x)| \leq \tan \alpha \text{Re}(Sx, x),$$

for all $x \in \text{dom}(S)$. In particular, if $\alpha = 0$, then $(Sx, x) \geq 0$ for all $x \in \text{dom}(S)$, i.e., S is symmetric and nonnegative operator. *In the sequel we will use the word "sectorial" only for sectorial operators and sectorial sesquilinear forms with vertex at the origin. In addition, if semi-angle of sectorial operator S is α we will call S α -sectorial operator.* A linear operator S is called *m-sectorial*, if it is sectorial and m -accretive. If T is m - α -sectorial operator and if $\gamma \in (\alpha, \pi/2)$ then

$$\lambda \in \mathbb{C} \setminus \Theta(\gamma) \Rightarrow \|(T - \lambda I)^{-1}\| \leq \frac{1}{|\lambda| \sin(\gamma - \alpha)}, \quad (0.2)$$

and the one-parameter semigroup $U(t) = \exp(-tT)$, $t \geq 0$, admits a holomorphic contractive continuation into the interior of the sector $\Theta(\pi/2 - \alpha)$ [28].

It is well-known that there is a one-to-one correspondence between closed densely defined sectorial forms and m -sectorial operators. This correspondence is given by the First and the Second Representations Theorems [28]. We will denote by $T[u, v]$ the closed form associated with m -sectorial extension T and by $\mathcal{D}[T]$ its domain.

In the present paper we continue to study m -accretive extensions of a densely defined closed sectorial operator S . It is well known [28], that S admits at least one m -sectorial extension S_F , the Friedrichs extension, which is associated with the closure of sesquilinear form (Sf, g) , $f, g \in \text{dom}(S)$. In [7, 8, 9, 10, 11, 13], the boundary triplets methods have been applied for a description of all m -accretive, m -sectorial extensions, and their resolvents for sectorial operators S satisfying condition

$$\text{dom}(S^*) \subseteq \text{D}[S_N], \quad (0.3)$$

where S_N is “extremal” m -sectorial extension of S , called the Kreĭn-von Neumann extension [7, 8]. Such extension is an analog of the “soft” (“the Kreĭn”, “the Kreĭn-von Neumann”) *nonnegative* selfadjoint extension of a nonnegative symmetric operator, discovered by M.G. Kreĭn in [30, 31]. Recall that S is called nonnegative if $(Sf, f) \geq 0$ for all $f \in \text{dom}(S)$. Observe that condition (0.3) holds true if for sectorial S the equality $\text{dom}(S_F^*) + \text{dom}(S_N^*) = \text{dom}(S^*)$ is satisfied. The latter occurs, for instance, if S is coercive, i.e., $\text{Re}(Sf, f) \geq m\|f\|^2$ for all $f \in \text{dom}(S)$, where $m > 0$.

In our recent paper [15] in the general case of an *arbitrary* closed densely defined sectorial operator S we propose a new approach for the problem of parametrization of all m -accretive extensions. Our method is applicable, in particular, for sectorial operator S , having a unique m -sectorial extension ($S_F = S_N$). In this paper, assuming $S_F \neq S_N$, we apply our method for a description of all m -sectorial extensions.

Let A be a densely defined closed symmetric operator in \mathfrak{H} . Extensions \tilde{A} of A possessing property

$$A \subset \tilde{A} \subset A^*$$

are called *quasi-selfadjoint (proper, intermediate)* extensions of S . The problem of existence and description of all quasi-selfadjoint m -accretive extensions of a nonnegative symmetric operator via linear-fractional transformation has been solved in [16] and via abstract boundary conditions in [35, 29, 5, 23, 22]. We refer on this matter to the survey [18] where one can find information about various approaches to the extension problem of nonnegative symmetric operators. Notice that in [14], developing the method proposed in [17], an intrinsic parametrization of domains of all m -accretive and m -sectorial quasi-selfadjoint extensions of nonnegative A have been obtained.

In the present paper we use the approach of [15] for such extensions. Applications to nonnegative symmetric operator in a planar model of two-centers point interactions are given. In one-center point interaction planar model the corresponding nonnegative symmetric operator admits a unique nonnegative selfadjoint extension [1], [24], hence, the Friedrichs extension is unique among all quasi-selfadjoint m -accretive extensions [41] and all m -sectorial extensions [7]. In our paper [15] we described all m -accretive extensions for this case. In the case of two and more centers, the Friedrichs extension is a non-unique element of the set of all nonnegative selfadjoint extensions, therefore, there are non-selfadjoint m -accretive quasi-selfadjoint extensions and m -sectorial extensions.

1. PRELIMINARIES

1.1. Sectorial forms and operators. The basic definitions and results concerning sesquilinear forms can be found in [28]. If τ is a closed densely defined sectorial form in the Hilbert space \mathfrak{H} , then by the First Representation Theorem [30, 28], there exists a unique m -sectorial operator T in \mathfrak{H} associated with τ in the following sense: $(Tu, v) = \tau[u, v]$, for all $u \in \text{dom}(T)$ and for all $v \in \text{dom}(\tau)$. The adjoint operator T^* is associated with the adjoint form $\tau^*[u, v] := \overline{\tau[v, u]}$. The nonnegative selfadjoint operator T_R associated with the real part $\tau_R[u, v] := (\tau[u, v] + \tau^*[u, v]) / 2$ of the form τ and is called the *real part* of T . According to the Second Representation Theorem [28] the equality $\text{dom}(\tau) = \text{dom}(T_R^{1/2})$ holds. Moreover,

$$\tau[u, v] = ((I + iG)T_R^{\frac{1}{2}}u, T_R^{\frac{1}{2}}v), \quad u, v \in \text{dom}(\tau),$$

where G is a bounded selfadjoint operator in the subspace $\overline{\text{ran}(T_R)}$ and $\|G\| \leq \tan \alpha$ iff τ is α -sectorial. It follows that

$$\begin{aligned} \text{dom}(T) &= \{u \in \text{dom}(\tau) : (I + iG)T_R^{1/2}u \in \text{dom}(\tau)\}, \\ Tu &= T_R^{1/2}(I + iG)T_R^{1/2}u, \quad u \in \text{dom}(T). \end{aligned} \tag{1.1}$$

In the sequel we will use the following notations for a m -sectorial operator T :

$$D[T] \stackrel{\text{def}}{=} \text{dom}(T_R^{1/2}), \quad R[T] \stackrel{\text{def}}{=} \text{ran}(T_R^{1/2}).$$

Also, for a m -sectorial operator T we denote by

$$\hat{T} = T \upharpoonright \overline{\text{ran}(T)}, \quad \hat{T}_R = T_R \upharpoonright \overline{\text{ran}(T)}.$$

Equality (0.1) yields that $\ker(\hat{T}) = \ker(\hat{T}_R) = \{0\}$. From (1.1) it follows for $\lambda = -a + ib$, $a, b \in \mathbb{R}$, $a > 0$ (see [27, 8])

$$\begin{aligned} (T - \lambda I)^{-1} &= (T_R + aI)^{-1/2}(I + iG(\lambda))^{-1}(T_R + aI)^{-1/2}, \\ G(\lambda) &= T_R^{1/2}(T_R + aI)^{-1/2}GT_R^{1/2}(T_R + aI)^{-1/2} - b(T_R + aI)^{-1}. \end{aligned}$$

The latter equalities imply the following statement.

Proposition 1.1 ([8]). *If $T = T_R^{1/2}(I + iG)T_R^{1/2}$ is a m - α -sectorial operator in the Hilbert space \mathfrak{H} , and $\gamma \in (\alpha, \pi/2)$, then*

$$\begin{aligned} R[T] &= \left\{ f \in \mathfrak{H} : \sup_{x \in \text{dom}(T)} \frac{|(f, x)|^2}{\text{Re}(Tx, x)} < \infty \right\} = \\ &= \left\{ f \in \mathfrak{H} : \lim_{\substack{\lambda \rightarrow 0, \\ \lambda \in \mathbb{C} \setminus \Theta(\alpha)}} |((T - \lambda I)^{-1}f, f)| < \infty \right\}; \end{aligned} \tag{1.2}$$

$$\begin{aligned} \lim_{\substack{\lambda \rightarrow 0, \\ \lambda \in \mathbb{C} \setminus \Theta(\gamma)}} ((T - \lambda I)^{-1}f, g) &= \hat{T}^{-1}[f, g] \\ &= ((I + iG)^{-1}\hat{T}_R^{-1/2}f, \hat{T}_R^{-1/2}g), \quad f, g \in R[T]; \end{aligned} \tag{1.3}$$

$$\lim_{\substack{\lambda \rightarrow 0, \\ \lambda \in \mathbb{C} \setminus \Theta(\gamma)}} T_R^{1/2} (T - \lambda I)^{-1} T_R^{1/2} g = (I + iG)^{-1} g; \quad g \in D[T] \ominus \ker(T). \quad (1.4)$$

1.2. The Friedrichs and Kreĭn-von Neumann m-sectorial extensions. Let S be an α -sectorial operator. It is well known [28], that the form (Su, v) , $u, v \in \text{dom}(S)$ is closable. We will denote by $S[u, v]$ its closure. The domain of the form $S[u, v]$ is denoted by $D[S]$. With the closed form $S[u, v]$ is associated the maximal α -sectorial operator S_F , which is called the *Friedrichs extension* of S [28]. So $D[S] = D[S_F]$ and $S_F[u, v] = S[u, v]$ for all $u, v \in D[S]$. Let S_{FR} be the real part of S_F . Clearly, $D[S] = \text{dom}(S_{FR}^{1/2})$. We will use the representations

$$\begin{aligned} S[u, v] &= S_F[u, v] = ((I + iG_F)S_{FR}^{1/2}u, S_{FR}^{1/2}v), \quad u, v \in D[S] = \text{dom}(S_{FR}^{1/2}), \\ S_F f &= S_{FR}^{1/2}(I + iG_F)S_{FR}^{1/2}f, \quad f \in \text{dom}(S_F), \\ S_F^* g &= S_{FR}^{1/2}(I - iG_F)S_{FR}^{1/2}g, \quad g \in \text{dom}(S_F^*). \end{aligned}$$

It follows from the definition of the closure of the form (Su, v) , that

$$R[S_F] = \text{ran}(S_{FR}^{1/2}) = \left\{ f \in \mathfrak{H} : \sup_{\varphi \in \text{dom}(S)} \frac{|(f, \varphi)|^2}{\text{Re}(S\varphi, \varphi)} < \infty \right\}. \quad (1.5)$$

In the case of a nonnegative symmetric operator S ($\alpha = 0$) it was discovered by M.G. Kreĭn [30] that the set of all its nonnegative selfadjoint extensions has a minimal element (in the sense of associated closed quadratic forms). This minimal element S_N is defined in [30] by means of linear-fractional transformation. Another (equivalent) definitions of S_N are given in [3] and in [20]. If $\alpha \neq 0$, then the corresponding m-sectorial analog of such extremal extension also exists [7, 8] and can be defined similarly, see [7, 8, 13]. We preserve the same notation S_N and the name *Kreĭn-von Neumann extension* in the general case of non necessarily symmetric sectorial operator S . We notice that interesting applications of Kreĭn-von Neumann extension of nonnegative symmetric operator can be found in [19].

The domain of closed sesquilinear form associated with Kreĭn-von Neumann extension of α -sectorial operator S is given by (see [8])

$$D[S_N] = \left\{ u \in \mathfrak{H} : \sup_{\varphi \in \text{dom}(S)} \frac{|(u, S\varphi)|^2}{\text{Re}(S\varphi, \varphi)} < \infty \right\}. \quad (1.6)$$

This is an analog of the formula established by T. Ando and K. Nishio in [3] for the case of nonnegative symmetric operator S ($\alpha = 0$).

Let

$$\mathfrak{N}_\lambda \stackrel{\text{def}}{=} \mathfrak{H} \ominus \text{ran}(S - \bar{\lambda}I)$$

be the defect subspace of a linear operator S . If S closed and densely defined, then

$$\mathfrak{N}_\lambda = \ker(S^* - \lambda I).$$

It is easy to see, that if \tilde{S} is an extension of S with nonempty resolvent set, then for all $\lambda, z \in \rho(\tilde{S}^*)$

$$(\tilde{S}^* - \lambda I)(\tilde{S}^* - zI)^{-1}\mathfrak{N}_\lambda = (I + (z - \lambda)(\tilde{S}^* - zI)^{-1})\mathfrak{N}_\lambda = \mathfrak{N}_z. \quad (1.7)$$

Note, that from (1.5) and (1.6) it follows that

$$D[S_N] \cap \mathfrak{N}_\lambda = R[S_F] \cap \mathfrak{N}_\lambda. \quad (1.8)$$

For the operators S_F , S_N , and for an arbitrary m -sectorial extension \tilde{S} of S the following relations are valid [7, 8]:

$$D[S] \cap \mathfrak{N}_\lambda = \{0\},$$

$$D[S_N] = D[S] \dot{+} (\mathfrak{N}_\lambda \cap D[S_N]), \quad \lambda \in \rho(S_F^*). \quad (1.9)$$

$$D[S] \subseteq D[\tilde{S}] \subseteq D[S_N], \quad R[S_N] \subseteq R[\tilde{S}] \subseteq R[S_F], \quad (1.10)$$

$$\tilde{S}[f, v] = S_N[f, v] \quad \forall f \in D[S], v \in D[\tilde{S}], \quad (1.11)$$

$$S_N[f, v] = (f, S^*v) \quad \forall f \in D[S], v \in \text{dom}(S^*) \cap D[S_N], \quad (1.12)$$

$$\text{dom}(S_F^*) = D[S] \cap \text{dom}(S^*). \quad (1.13)$$

If S is coercive, then

$$\begin{aligned} \text{dom}(S_N) &= \text{dom}(S) \dot{+} \ker(S^*), \quad S_N \upharpoonright \ker(S^*) = 0, \\ D[S_N] &= D[S] \dot{+} \ker(S^*). \end{aligned}$$

The operator S has a unique m -sectorial extension if and only if, for some $\lambda \in \rho(S_F^*)$ (then for all $\lambda \in \rho(S_F^*)$):

$$\sup_{x \in \text{dom}(S)} \frac{|(f_\lambda, x)|^2}{\text{Re}(Sx, x)} = \infty \quad \forall f_\lambda \in \mathfrak{N}_\lambda \setminus \{0\}. \quad (1.14)$$

From (1.5), (1.6), and (1.14) it follows that

$$\begin{aligned} S_N \neq S_F &\iff D[S_N] \cap \mathfrak{N}_\lambda \neq \{0\} \\ &\iff R[S_F] \cap \mathfrak{N}_\lambda \neq \{0\}, \quad \lambda \in \rho(S_F^*). \end{aligned} \quad (1.15)$$

Taking into account (1.15), (1.2), (1.3) we get for $\mu \in \mathbb{C} \setminus \Theta(\alpha)$

$$\varphi_\mu \in \mathfrak{N}_\mu \cap D[S_N] \iff \lim_{\substack{\lambda \rightarrow 0, \\ \lambda \in \mathbb{C} \setminus \Theta(\alpha)}} |((S_F^* - \lambda I)^{-1} \varphi_\mu, \varphi_\mu)| < \infty, \quad (1.16)$$

and for $\gamma \in (\alpha, \pi/2)$

$$\lim_{\substack{\lambda \rightarrow 0, \\ \lambda \in \mathbb{C} \setminus \Theta(\gamma)}} ((S_F^* - \lambda I)^{-1} \varphi_\mu, \psi_\mu) = ((I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \varphi_\mu, \hat{S}_{FR}^{-1/2} \psi_\mu),$$

$$\varphi_\mu, \psi_\mu \in \mathfrak{N}_\mu \cap D[S_N].$$

Fix $z \in \rho(S_F^*)$ and define a linear manifold \mathfrak{L} :

$$\mathfrak{L} \stackrel{\text{def}}{=} D[S] \dot{+} \mathfrak{N}_z, \quad z \in \rho(S_F^*). \quad (1.17)$$

Then \mathfrak{L} does not depend on the choice of $z \in \rho(S_F^*)$ [15] and, clearly, $\text{dom}(S^*) \subset \mathfrak{L}$. We will denote by $\mathcal{P}_{z,F}$ and \mathcal{P}_z the skew projectors in \mathfrak{L} onto $D[S]$ and \mathfrak{N}_z , corresponding to the decomposition (1.17). If $z = i$, these projectors we will denote by \mathcal{P}_F and \mathcal{P}_i , respectively.

Let us consider the form $\hat{S}_z[u, v]$ defined on the linear manifold \mathfrak{L}

$$\hat{S}_z[u, v] = S[\mathcal{P}_{\bar{z}, F}u, \mathcal{P}_{\bar{z}, F}v] - z(\mathcal{P}_{\bar{z}, F}u, \mathcal{P}_{\bar{z}, F}v), \quad \forall z \in \mathbb{C} \setminus \Theta(\alpha).$$

The following relations have been established in [8]:

$$D[S_N] = \left\{ u \in \mathfrak{L} : \lim_{\substack{z \rightarrow 0 \\ z \in \mathbb{C} \setminus \Theta(\alpha)}} |\hat{S}_z[u]| < \infty \right\},$$

$$S_N[u, v] = \lim_{\substack{z \rightarrow 0 \\ z \in \mathbb{C} \setminus \Theta(\gamma)}} \hat{S}_z[u, v], \quad u, v \in D[S_N], \quad \gamma \in (\alpha; \pi/2),$$

$$S_N[u, v] = \left((I + iG_F) \left(S_{FR}^{1/2} \mathcal{P}_{z, F}u + z(I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \mathcal{P}_z u \right), \right. \\ \left. \left(S_{FR}^{1/2} \mathcal{P}_{z, F}v + z(I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \mathcal{P}_z v \right) \right), \quad u, v \in D[S_N]. \quad (1.18)$$

1.3. Boundary triplets and abstract boundary conditions for quasi-selfadjoint extensions of nonnegative symmetric operator. Let A be a closed densely defined symmetric operator in \mathfrak{H} . Recall the definition of a boundary triplet (boundary value space) [25] for A^* .

Definition 1.2. A triplet $\{\mathcal{H}, \Gamma_1, \Gamma_0\}$ is called boundary triplet of A^* if \mathcal{H} is a Hilbert space and Γ_0, Γ_1 are bounded linear operators from the Hilbert space $H_+ = \text{dom}(S^*)$ with the graph norm into \mathcal{H} such that the map $\vec{\Gamma} = \langle \Gamma_0, \Gamma_1 \rangle$ is a surjection from H_+ onto $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$ and the Green identity holds:

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}} \quad \forall f, g \in H_+. \quad (1.19)$$

In the sequel for descriptions of extensions in terms of the abstract boundary conditions the *linear relations* will be used. One can find basic notions, and properties related to these objects in, for instance, [4, 39, 25, 22, 13]. The formulas

$$\text{dom}(\tilde{A}) = \left\{ u \in \text{dom}(A^*) : \vec{\Gamma}u \in \tilde{\mathbf{T}} \right\}, \quad \tilde{A} = A^* \upharpoonright \text{dom}(\tilde{A}) \quad (1.20)$$

give a one-to-one correspondence between all quasi-selfadjoint extensions \tilde{A} of A ($A \subset \tilde{A} \subset A^*$) and all linear relations $\tilde{\mathbf{T}}$ in \mathcal{H} . Moreover $\tilde{A}^* \leftrightarrow \tilde{\mathbf{T}}^*$. Therefore, an extension \tilde{A} is a selfadjoint one if and only if the relation $\tilde{\mathbf{T}}$ is a selfadjoint in \mathcal{H} .

As it was shown in [21, 22] the operators A_0, A_1 defined as follows

$$A_k = A^* \upharpoonright \text{Ker } \Gamma_k, \quad k = 0, 1$$

are mutually transversal selfadjoint extensions of A , i.e.,

$$\text{dom}(A^*) = \text{dom}(A_0) + \text{dom}(A_1).$$

The function $\Gamma_0(\lambda) := (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda)^{-1}$ [21] is the γ -field corresponding to A_0 [32, 33], i.e.,

$$\text{ran}(\Gamma_0(\lambda)) = \mathfrak{N}_\lambda,$$

$$\Gamma_0(\lambda) = \Gamma(z) + (\lambda - z)(A_0 - zI)^{-1}\Gamma_0(z).$$

Note that as a consequence of (1.19) one can obtain the equality

$$\Gamma_0(\bar{\lambda}) = (\Gamma_1(A_0 - \lambda I)^{-1})^*. \quad (1.21)$$

V. Derkach and M. Malamud [21, 22] define the Weyl function $M_0(\lambda)$ by the equality

$$M_0(\lambda) = \Gamma_1\Gamma_0(\lambda). \quad (1.22)$$

The Nevanlinna class operator valued function M_0 is Kreĭn-Langer Q -function [32, 33], and the following identity

$$M_0(\lambda) - M_0(z) = (\lambda - z)\Gamma_0^*(\bar{z})\Gamma_0(\lambda) \quad (1.23)$$

holds. In terms of boundary triplet the connection between a quasi-selfadjoint extension $\tilde{A}_{\tilde{\mathbf{T}}}$ defined by relations (1.20) and its resolvent is given by the Kreĭn resolvent formula

$$\begin{aligned} (\tilde{A}_{\tilde{\mathbf{T}}} - \lambda I)^{-1} &= (A_0 - \lambda I)^{-1} + \Gamma_0(\lambda) \left(\tilde{\mathbf{T}} - M_0(\lambda) \right)^{-1} \Gamma_0^*(\bar{\lambda}), \\ \lambda &\in \rho(A_0) \cap \rho(\tilde{A}_{\tilde{\mathbf{T}}}). \end{aligned} \quad (1.24)$$

The following theorem has been established by V. Derkach and M. Malamud (see [21, 22, 23, 34]).

Theorem 1.3. *Let A be a closed nonnegative symmetric operator and let $\{\mathcal{H}, \Gamma_1, \Gamma_0\}$ be a boundary triplet of A^* such that $A_0 = A_F (= A^* \upharpoonright \text{Ker } \Gamma_0)$. Then A has a non-unique nonnegative selfadjoint extension if and only if*

$$\mathcal{D}_0 = \left\{ h \in \mathcal{H} : \lim_{x \uparrow 0} (M_0(x)h, h)_{\mathcal{H}} < \infty \right\} \neq \{0\},$$

and the quadratic form

$$\tau[h] = \lim_{x \uparrow 0} (M_0(x)h, h)_{\mathcal{H}}, \quad \mathcal{D}[\tau] = \mathcal{D}_0$$

is bounded from below. Define by $M_0(0)$ the selfadjoint linear relation in \mathcal{H} associated with τ . Then the Kreĭn-von Neumann extension A_N can be defined by the boundary condition

$$\text{dom}(A_N) = \{u \in \text{dom}(A^*) : \langle \Gamma_0 u, \Gamma_1 u \rangle \in M_0(0)\}.$$

The relation $M_0(0)$ is also the strong resolvent limit of $M_0(x)$ when $x \rightarrow -0$. Moreover, A_0 and A_N are disjoint iff $\overline{\mathcal{D}_0} = \mathcal{H}$ and transversal iff $\mathcal{D}_0 = \mathcal{H}$. In there is a one-to-one correspondence given by (1.20) between m -accretive extensions $\tilde{A}_{\tilde{\mathbf{T}}}$ and m -accretive linear relations $\tilde{\mathbf{T}}$ satisfying the condition

$$\text{dom}(\tilde{\mathbf{T}}) \subseteq \mathcal{D}_0, \quad \text{Re}(\tilde{\mathbf{T}}x, x) \geq \tau[x], \quad x \in \text{dom}(\tilde{\mathbf{T}}). \quad (1.25)$$

The extension $\tilde{A}_{\tilde{\mathbf{T}}}$ is m - α -sectorial iff the form

$$(\tilde{\mathbf{T}}x, y) - \tau[x, y]$$

is α -sectorial.

2. ABSTRACT BOUNDARY CONDITIONS FOR m -ACCRETIVE EXTENSIONS OF SECTORIAL OPERATORS

Next, we recall some definitions and results established in [15]. A sesquilinear form

$$\tau[u, v] \stackrel{\text{def}}{=} S_{FR}[\mathcal{P}_{-1,F}u, \mathcal{P}_{-1,F}v] + (\mathcal{P}_{-1}u, \mathcal{P}_{-1}v), \quad u, v \in \mathfrak{L}$$

is a nonnegative and closed [15] in the Hilbert space \mathfrak{H} . So, we can consider the linear manifold \mathfrak{L} as a Hilbert space with the inner product

$$(u, v)_\tau = \tau(u, v) + (u, v)_\mathfrak{H}.$$

Definition 2.1 ([15]). *A pair $\{\mathcal{H}, \Gamma\}$ is called boundary pair of S , if \mathcal{H} is a Hilbert space and $\Gamma \in \mathbf{L}(\mathfrak{L}, \mathcal{H})$ is such that $\ker(\Gamma) = \mathbf{D}[S]$, $\text{ran}(\Gamma) = \mathcal{H}$.*

Let

$$\gamma(\lambda) = (\Gamma \upharpoonright \mathfrak{N}_\lambda)^{-1}, \quad \lambda \in \rho(S_F^*).$$

Then $\gamma(\lambda) \in \mathbf{L}(\mathcal{H}, \mathfrak{H})$ for all $\lambda \in \rho(S_F^*)$. The operator-function $\gamma(\lambda)$ is called γ -field of the operator S associated with the boundary pair $\{\mathcal{H}, \Gamma\}$. Clearly, $\gamma(\lambda)$ maps \mathcal{H} onto \mathfrak{N}_λ . Hence $S^*\gamma(\lambda) = \lambda\gamma(\lambda)$ and

$$\ker(\gamma^*(\lambda)) = \text{ran}(S - \bar{\lambda}I)$$

The following relations are valid:

$$\gamma(\lambda) = \gamma(z) + (\lambda - z)(S_F^* - \lambda I)^{-1}\gamma(z), \quad (2.1)$$

$$\mathcal{P}_{\lambda,F}u = u - \gamma(\lambda)\Gamma u, \quad u \in \mathfrak{L},$$

$$\mathcal{P}_F\gamma(\lambda)e = (\lambda - i)(S_F^* - \lambda I)^{-1}\gamma(i)e, \quad \mathcal{P}_i\gamma(\lambda)e = \gamma(i)e, \quad e \in \mathcal{H}.$$

Define on \mathfrak{L} one more sesquilinear form $l[u, v]$:

$$l[u, v] = S_F[\mathcal{P}_Fu, \mathcal{P}_Fv] - i(\mathcal{P}_iu, \mathcal{P}_Fv) - i(\mathcal{P}_Fu, \mathcal{P}_iv) - i(\mathcal{P}_iu, \mathcal{P}_iv). \quad (2.2)$$

Due to the equality

$$\text{Re } l[u] = \text{Re } S[\mathcal{P}_Fu] = \left\| S_{FR}^{1/2} \mathcal{P}_Fu \right\|^2, \quad u \in \mathfrak{L},$$

the form $l[u, v]$ is accretive. Moreover,

$$\inf_{\varphi \in \mathbf{D}[S]} \{\text{Re } l[u - \varphi]\} = 0, \quad \forall u \in \mathfrak{L},$$

and $l[\varphi, v] = (\varphi, S^*v)$ for all $\varphi \in \mathbf{D}[S]$, $v \in \text{dom}(S^*)$.

Relations (1.18) and (2.2) imply the following representation of the form $S_N[\cdot, \cdot]$:

$$\begin{aligned} S_N[u, v] &= l[u, v] \\ &+ \left[i(\gamma(i)\Gamma u, \gamma(i)\Gamma v) + \left((I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i)\Gamma u, \hat{S}_{FR}^{-1/2} \gamma(i)\Gamma v \right) \right] \\ &+ 2i \left((I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i)\Gamma u, S_{FR}^{1/2} \mathcal{P}_Fv \right), \quad u, v \in \mathbf{D}[S_N]. \end{aligned} \quad (2.3)$$

Definition 2.2 ([15]). *The triplet $\{\mathcal{H}, G, \Gamma\}$ is called boundary triplet for the operator S^* if $\{\mathcal{H}, \Gamma\}$ is a boundary pair for S and $G: \text{dom}(S^*) \rightarrow \mathcal{H}$ is a linear operator such that the relation*

$$l^*[u, v] = (S^*u, v) - (Gu, \Gamma v)_{\mathcal{H}}, \quad \forall u \in \text{dom}(S^*), \forall v \in \mathfrak{L} \quad (2.4)$$

is valid.

It is shown in [15] that there exists a unique operator $G: \text{dom}(S^*) \rightarrow \mathcal{H}$ such that, (2.4) holds and, moreover,

$$Gu = \gamma^*(i)(S^* - iI)u.$$

Next, we define operator-functions $\mathcal{Q}(\lambda) \in \mathbf{L}(\mathcal{H})$, $\mathcal{G}(\lambda) \in \mathbf{L}(\mathfrak{H}, \mathcal{H})$, $\Phi(\lambda) \in \mathbf{L}(\mathfrak{H}, \mathfrak{H})$, $q(\lambda) \in \mathbf{L}(\mathcal{H}, \mathfrak{H})$, $\lambda \in \rho(S_F)$ associated with the boundary triplet for the operator S^* , see [15]:

$$\begin{aligned} \mathcal{Q}(\lambda) &\stackrel{\text{def}}{=} G\gamma(\lambda), \quad q(\lambda) \stackrel{\text{def}}{=} (G(S_F^* - \bar{\lambda}I)^{-1})^*, \\ \mathcal{G}(\lambda) &\stackrel{\text{def}}{=} \left(S_{FR}^{1/2}\mathcal{P}_F\gamma(\bar{\lambda})\right)^*, \quad \Phi(\lambda) \stackrel{\text{def}}{=} \left(S_{FR}^{1/2}(S_F^* - \bar{\lambda}I)^{-1}\right)^*. \end{aligned}$$

The following identities are valid [15]:

$$\begin{aligned} \mathcal{Q}(\lambda) &= \gamma^*(i)(S_F^* - iI)\gamma(\lambda) = (\lambda - i)\gamma^*(i)\gamma(\lambda), \\ \Phi(\lambda) - \Phi(z) &= (\lambda - z)(S_F - \lambda I)^{-1}\Phi(z) = (\lambda - z)(S_F - zI)^{-1}\Phi(\lambda), \\ \mathcal{G}(\lambda) - \mathcal{G}(z) &= (\lambda - z)\gamma^*(\bar{z})\Phi(\lambda), \\ q(\lambda) - q(z) &= (\lambda - z)(S_F - \lambda I)^{-1}q(z), \\ \mathcal{Q}(\lambda) - \mathcal{Q}(z) &= (\lambda - z)q^*(\bar{\lambda})\gamma(z). \end{aligned} \quad (2.5)$$

Observe that the function $\mathcal{Q}(\lambda)$ is an analog of the Weyl function (1.22) corresponding to a boundary triplet of the adjoint to a symmetric operator, while $q(\lambda)$ is an analog of the function in (1.21).

Let L be a linear operator in \mathfrak{L} defined as follows:

$$\begin{aligned} \text{dom}(L) &= \text{dom}(S_F) \dot{+} \mathfrak{N}_i, \\ L(u_F + u_i) &= S_F u_F - i u_i, \quad u_F \in \text{dom}(S_F), u_i \in \mathfrak{N}_i. \end{aligned} \quad (2.6)$$

Then L is closed, and

$$\begin{aligned} (Lu, \varphi) &= l[u, \varphi] \quad \forall u \in \text{dom}(L), \varphi \in \mathbf{D}[S], \\ \ker(L - \lambda I) &= \text{ran}(q(\lambda)) \quad \forall \lambda \in \rho(S_F), \\ \text{dom}(L) &= \text{dom}(S_F) \dot{+} \text{ran}(q(\lambda)) \quad \forall \lambda \in \rho(S_F). \end{aligned}$$

Definition 2.3 ([15]). *Let S be a densely defined sectorial operator and let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for S . A triplet $\{\mathcal{H}, G_*, \Gamma\}$ is called a boundary triplet for L if $G_*: \text{dom}(L) \rightarrow \mathcal{H}$ is a linear operator such that*

$$l[u, v] = (Lu, v) - (G_*u, \Gamma v)_{\mathcal{H}}, \quad \forall u \in \text{dom}(L), \forall v \in \mathfrak{L}.$$

The operator G_* is uniquely defined [15] and, moreover, for each $\lambda \in \rho(S_F)$

$$\begin{aligned} G_* f &= \gamma^*(\bar{\lambda})(S_F - \lambda I)f, \quad f \in \text{dom}(S_F), \\ G_* q(\lambda)e &= \mathcal{Q}^*(\bar{\lambda})e, \quad e \in \mathcal{H}. \end{aligned} \quad (2.7)$$

Thus, given a boundary pair $\{\mathcal{H}, \Gamma\}$ for an operator S , the boundary triplets corresponding to it are $\{\mathcal{H}, G, \Gamma\}$ for S^* and $\{\mathcal{H}, G_*, \Gamma\}$ for L , and we have the abstract Green formula

$$(Lu, v) - (u, S^*v) = (G_*u, \Gamma v)_{\mathcal{H}} - (\Gamma u, Gv)_{\mathcal{H}}, \quad \forall u \in \text{dom}(L), \quad \forall v \in \text{dom}(S^*).$$

Let \tilde{S} be an m -accretive extension of S . The following inclusions are established in [15]:

$$\begin{aligned} \text{dom}(\tilde{S}) &\subseteq \mathfrak{L}, \\ \tilde{S}u + \lambda \mathcal{P}_\lambda u &\in \text{ran}(S_F^{1/2}) (= \text{R}[S_F]), \quad \lambda \in \rho(S_F^*). \end{aligned} \quad (2.8)$$

The next two theorems follow from (2.8).

Theorem 2.4 ([15]). *Let S be a densely defined closed sectorial operator. Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for S and $\{\mathcal{H}, G, \Gamma\}$ be a corresponding boundary triplet for S^* . If \tilde{S} is an m -accretive extension of S , then there exist linear operators*

$$Z : \text{dom}(\tilde{S}) \rightarrow \mathcal{H} \text{ and } X : \text{dom}(\tilde{S}) \rightarrow \overline{\text{ran}(S_F)},$$

such that:

- 1) $\text{dom}(S) \subseteq \ker(Z)$;
- 2) $(\tilde{S}u, v) = l[u, v] + (Zu, \Gamma v)_{\mathcal{H}} + 2(X\Gamma u, S_F^{1/2}\mathcal{P}_F v)$, $\forall u \in \text{dom}(\tilde{S}), v \in \mathfrak{L}$
- 3) $\mathbf{Z} = \{(\Gamma u, Zu), u \in \text{dom}(\tilde{S})\}$ — is an m -accretive linear relation in \mathcal{H} ;
- 4) $\|Xe\|^2 \leq \text{Re}(\mathbf{Z}(e), e)_{\mathcal{H}}$ for all $e \in \text{dom}(\mathbf{Z}) = \Gamma \text{dom}(\tilde{S})$

Theorem 2.5 ([15]). *There is a bijective correspondence between all m -accretive extensions \tilde{S} of S and all pairs $\langle \mathbf{Z}, X \rangle$, where \mathbf{Z} is an m -accretive linear relation in \mathcal{H} and $X : \text{dom}(\mathbf{Z}) \rightarrow \overline{\text{ran}(S_F)}$ is a linear operator such that:*

$$\|Xe\|^2 \leq \text{Re}(\mathbf{Z}(e), e)_{\mathcal{H}} \quad \forall e \in \text{dom}(\mathbf{Z}). \quad (2.9)$$

This correspondence is given by the boundary conditions for the domain and the action of \tilde{S} as follows: for all $\text{Re } \lambda < 0$

$$\begin{aligned} \text{dom}(\tilde{S}) &= \left\{ u \in \mathfrak{L} : \begin{array}{l} 1) \quad u - (q(\lambda) - 2\Phi(\lambda)X)\Gamma u \in \text{dom}(S_F); \\ 2) \quad G_*(u + 2\Phi(\lambda)X\Gamma u) \in (\mathbf{Z} + 2\mathcal{G}(\lambda)X)\Gamma u \end{array} \right\}, \\ \tilde{S}u &= S_F(u - (q(\lambda) - 2\Phi(\lambda)X)\Gamma u) + \lambda(q(\lambda) - 2\Phi(\lambda)X)\Gamma u. \end{aligned} \quad (2.10)$$

Set

$$\mathbf{W}(\lambda) := \mathbf{Z} - \mathcal{Q}^*(\bar{\lambda}) + 2\mathcal{G}(\lambda), \quad \lambda \in \rho(S_F). \quad (2.11)$$

Then

- 1) a number $\lambda \in \rho(S_F)$ is a regular point of \tilde{S} if and only if

$$\mathbf{W}^{-1}(\lambda) \in \mathbf{L}(\mathcal{H}),$$

and,

$$(\tilde{S} - \lambda I)^{-1} = (S_F - \lambda I)^{-1} + (q(\lambda) - 2\Phi(\lambda)X)\mathbf{W}^{-1}(\lambda)\gamma^*(\bar{\lambda}), \quad (2.12)$$

$$\text{dom}(\tilde{S}) = \left(I + (q(\lambda) - 2\Phi(\lambda)X)\mathbf{W}^{-1}(\lambda)\gamma^*(\bar{\lambda})(S_F - \lambda I) \right) \text{dom}(S_F), \quad (2.13)$$

$$\tilde{S}u = (S_F - \lambda I)f + \lambda u \quad (2.14)$$

for

$$u = \left(I + (q(\lambda) - 2\Phi(\lambda)X)\mathbf{W}^{-1}(\lambda)\gamma^*(\bar{\lambda})(S_F - \lambda I) \right) f, \quad f \in \text{dom}(S_F), \quad (2.15)$$

2) a number $\lambda \in \rho(S_F)$ is an eigenvalue of \tilde{S} if and only if

$$\ker(\mathbf{W}(\lambda)) \neq \{0\},$$

and,

$$\ker(\tilde{S} - \lambda I) = (q(\lambda) - 2\Phi(\lambda)X) \ker(\mathbf{W}(\lambda)).$$

Remark 2.6. Relations (2.10) remain valid for all $\lambda \in \rho(\tilde{S}) \cap \rho(S_F)$. The resolvent formula (2.12) is an analog of the resolvent formula (1.24).

Let S be a densely defined closed sectorial operator. Define for all $z \in \mathbb{C}$, $\text{Re } z \leq 0$ a linear operator S_z as follows [9, 10]:

$$\begin{aligned} \text{dom}(S_z) &= \text{dom}(S) \dot{+} \mathfrak{N}_z, \\ S_z h &= S\varphi - z\varphi_z, \quad h = \varphi + \varphi_z \in \text{dom}(S_z). \end{aligned} \quad (2.16)$$

Proposition 2.7 ([9, 10]). *The operator S_z is m -accretive extension of S .*

Proof. Proposition has been proved in [9, 10] for $\text{Re } z < 0$. Let us prove the statement for $z = ix$, $x \in \mathbb{R}$. Let $g = \varphi + \varphi_{ix}$, $\varphi \in \text{dom}(S)$, $\varphi_{ix} \in \mathfrak{N}_{ix}$. Then

$$\begin{aligned} (S_{ix}g, g) &= (S\varphi - ix\varphi_{ix}, \varphi + \varphi_{ix}) \\ &= (S\varphi, \varphi) - ix\|\varphi_{ix}\|^2 - 2i\text{Im}(ix(\varphi_{ix}, \varphi)) \end{aligned}$$

Hence $\text{Re}(Sg, g) = \text{Re}(S\varphi, \varphi) \geq 0$ for all $g \in \text{dom}(S_{ix})$. Furthermore, one can verify that

$$\begin{cases} \text{dom}(S_{ix}^*) = (S_F^* - ixI)^{-1}(S + ixI) \text{dom}(S) \dot{+} \mathfrak{N}_{ix}, \\ S_{ix}^* ((S_F^* - ixI)^{-1}(S + ixI)f + \varphi_{ix}) = S_F^* (S_F^* - ixI)^{-1}(S + ixI)f + ix\varphi_{ix}, \\ f \in \text{dom}(S), \quad \varphi_{ix} \in \mathfrak{N}_{ix} \end{cases}$$

and

$$\text{Re}(S_{ix}^* h, h) = \text{Re} \left(S_F^* (S_F^* - ixI)^{-1}(S + ixI)f, (S_F^* - ixI)^{-1}(S + ixI)f \right) \geq 0.$$

for

$$h = (S_F^* - ixI)^{-1}(S + ixI)f + \varphi_{ix}, \quad f \in \text{dom}(S), \quad \varphi_{ix} \in \mathfrak{N}_{ix}.$$

This means that S_{ix}^* is accretive. Thus, S_{ix} and S_{ix}^* are accretive. It follows that S_{ix} is m -accretive. \square

Note that in general from (2.16) it follows for $\operatorname{Re} z \leq 0$ that

$$\begin{aligned}\operatorname{dom}(S_z^*) &= \{g \in \operatorname{dom}(S^*) : (S^* + \bar{z}I)g \in \operatorname{ran}(S - \bar{z}I)\}, \\ S_z^* &= S^* \upharpoonright \operatorname{dom}(S_z^*).\end{aligned}$$

In addition, for the boundary operators in the boundary triplets in Definitions 2.2 and 2.3, the equalities are valid

$$\ker(G) = \operatorname{dom}(S_i^*), \quad \ker(G_*) = \operatorname{dom}(S_i).$$

Remark 2.8. It is proved in [10, 12] that

1) for each $\gamma \in [0, \pi/2)$ the equalities are valid:

$$\begin{aligned}\text{s-R-lim}_{\substack{z \rightarrow 0 \\ -z \in \Theta(\gamma)}} S_z &= S_N, & \text{s-R-lim}_{\substack{z \rightarrow \infty \\ -z \in \Theta(\gamma)}} S_z &= S_F,\end{aligned}$$

where s-R-lim is the strong resolvent limit [28];

2) the following conditions are equivalent:

- (a) S_z is m -sectorial operator for one (then for all) z , $\operatorname{Re} z < 0$;
- (b) $\operatorname{dom}(S^*) \subset \operatorname{dom}(S_N)$, where S_N is the Kreĭn–von Neumann extension of S .

Next we give expressions for pairs $\langle \mathbf{Z}_z, X_z \rangle$ corresponding to S_z , $\operatorname{Re} z \leq 0$ in accordance with Theorem 2.4.

Proposition 2.9. \mathbf{Z}_z is the graph of the operator $Z_z = -\mathcal{Q}(z)$, $\operatorname{dom}(Z_z) = \mathcal{H}$ and $X_z = -\mathcal{G}^*(\bar{z})$. In addition, for $u \in \operatorname{dom}(S_z)$, $v \in \mathfrak{L}$

$$(S_z u, v) = l[u, v] - (\mathcal{Q}(z)\Gamma u, \Gamma v)_{\mathcal{H}} - 2(\mathcal{G}^*(\bar{z})\Gamma u, S_{FR}^{1/2}\mathcal{P}_F v). \quad (2.17)$$

Proof. Define for $u \in \operatorname{dom}(S_z)$

$$\begin{aligned}Z_z u &:= \gamma^*(i)(S_z + iI)u, \\ M_z u &:= \frac{1}{2} \left(S_{FR}^{-1/2}(S_z u + i\mathcal{P}_i u) - (I + iG_F)S_{FR}^{1/2}\mathcal{P}_F u \right).\end{aligned} \quad (2.18)$$

Observe that from (2.18) one obtains the inclusions $\operatorname{dom}(S) \subseteq \ker(Z)$ and $\operatorname{dom}(S) \subset \ker(M_z)$. In addition, due to definition of \mathcal{L} (1.17), Definition 2.1 of a boundary pair, and (2.16), one obtains the equality

$$\Gamma \operatorname{dom}(S_z) = \mathcal{H}.$$

According to the proof of Theorem 2.4 (see [15]), the relations

$$\mathbf{Z}_z = \{ \langle \Gamma u, Z_z u \rangle, X_z \Gamma u = M_z u, u \in \operatorname{dom}(S_z) \}$$

hold. Then, taking into account that $u = \gamma(z)\Gamma u$ and relations (2.4), (2.5), (2.16), we have

$$\begin{aligned}Z_z u &= \gamma^*(i)(S_z + iI)\gamma(z)\Gamma u = \gamma^*(i)(-z\gamma(z)\Gamma u + i\gamma(z)\Gamma u) = \\ &= -(z - i)\gamma^*(i)\gamma(z)\Gamma u = -\mathcal{Q}(z)\Gamma u.\end{aligned}$$

Let $\Gamma u = e$, then $u = \varphi + \gamma(z)e$, $\varphi \in \operatorname{dom}(S)$, and

$$X_z \Gamma u = M_z u = M_z \gamma(z)e =$$

$$\begin{aligned}
&= \frac{1}{2} \left(S_{FR}^{-1/2} (S_z \gamma(z)e + iP_i \gamma(z)e) - (I + iG_F) S_{FR}^{1/2} \mathcal{P}_F \gamma(z)e \right) = \\
&= \frac{1}{2} \left(S_{FR}^{-1/2} (-z \gamma(z)e + i \gamma(i)e) - (I + iG_F) S_{FR}^{1/2} \mathcal{P}_F \gamma(z)e \right) = \\
&= \frac{1}{2} \left(S_{FR}^{-1/2} (-S^* \gamma(z)e + S^* \gamma(i)e) - (I + iG_F) S_{FR}^{1/2} \mathcal{P}_F \gamma(z)e \right) = \\
&= \frac{1}{2} \left(-S_{FR}^{-1/2} S_F^* \mathcal{P}_F \gamma(z)e - (I + iG_F) S_{FR}^{1/2} \mathcal{P}_F \gamma(z)e \right) = \\
&= \frac{1}{2} \left(-(I - iG_F) S_{FR}^{1/2} \mathcal{P}_F \gamma(z)e - (I + iG_F) S_{FR}^{1/2} \mathcal{P}_F \gamma(z)e \right) = \\
&= -S_{FR}^{1/2} \mathcal{P}_F \gamma(z)e = -\mathcal{G}^*(\bar{z}) \Gamma u.
\end{aligned}$$

Equality (2.17) follows from Theorem 2.4. \square

3. m -SECTORIAL EXTENSIONS

By Theorem 2.5, there is a bijective correspondence between all m -accretive extensions \tilde{S} of S and all pairs $\langle \mathbf{Z}, X \rangle$ satisfying condition (2.9). Our main goal is to establish additional conditions which guarantee that corresponding m -accretive extension \tilde{S} is sectorial.

Next, we will need the following auxiliary result:

Lemma 3.1. 1) *If T is a m -accretive operator and $\beta \in (0, \pi/2)$, then:*

$$\lim_{\substack{z \rightarrow 0, \\ \pi/2 + \beta \leq |\arg z| \leq \pi}} z(T - zI)^{-1}h = \begin{cases} -h, & h \in \ker(T) \\ 0, & h \in \overline{\operatorname{ran}(T)} \end{cases}. \quad (3.1)$$

2) *If T is m - α -sectorial and $\beta \in (\alpha, \pi/2)$, then*

$$\lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\beta)}} z(T - zI)^{-1}h = \begin{cases} -h, & h \in \ker(T) \\ 0, & h \in \overline{\operatorname{ran}(T)} \end{cases}. \quad (3.2)$$

Proof. 1) Clearly

$$h \in \ker(T) \Rightarrow (T - zI)^{-1}h = -\frac{h}{z} \quad \text{for all } z \in \rho(T) \setminus \{0\}.$$

Therefore

$$\lim_{\substack{z \rightarrow 0, \\ \pi/2 + \beta \leq |\arg z| \leq \pi}} z(T - zI)^{-1}h = -h.$$

Now let, $h \in \operatorname{ran}(T)$. Then $h = T\varphi$, $\varphi \in \operatorname{dom}(T)$ and

$$\begin{aligned}
z(T - zI)^{-1}h &= z(T - zI)^{-1}T\varphi = \\
&= z(T - zI)^{-1}(T - zI + zI)\varphi = z\varphi - z^2(T - zI)^{-1}\varphi.
\end{aligned}$$

Taking into account that

$$\|(T - zI)^{-1}\| \leq \frac{1}{|\operatorname{Re} z|}, \quad \operatorname{Re} z < 0,$$

and $|\operatorname{Re} z| \geq |z| \sin \beta$ for $\pi/2 + \beta \leq |\arg z| \leq \pi$, we get for all $\varphi \in \operatorname{dom}(T)$ that

$$\lim_{\substack{z \rightarrow 0, \\ \pi/2 + \beta \leq |\arg z| \leq \pi}} z(T - zI)^{-1}T\varphi = 0.$$

Further, since $\operatorname{ran}(T)$ is dense in $\overline{\operatorname{ran}(T)}$ and

$$\|z(T - zI)^{-1}\| \leq \frac{1}{\sin \beta}, \quad \pi/2 + \beta \leq |\arg z| \leq \pi,$$

then

$$\lim_{\substack{z \rightarrow 0, \\ \pi/2 + \beta \leq |\arg z| \leq \pi}} z(T - zI)^{-1}h$$

for all $h \in \overline{\operatorname{ran}(T)}$. Thus (3.1) is valid.

2) Relation (3.2) follows from (0.2). \square

Proposition 3.2. *Let S be a densely defined closed α -sectorial operator, $\gamma(z)$ its γ -field, corresponding to the boundary pair $\{\mathcal{H}, \Gamma\}$ of S . Suppose $S_F \neq S_N$. Then for all $e \in \mathcal{H}$ such that, $\gamma(\lambda)e \in \operatorname{D}[S_N]$:*

$$\lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\beta)}} z\gamma(z)e = 0,$$

where $\beta \in (0, \pi/2)$.

Proof. Let $\gamma(\lambda)e \in \operatorname{D}[S_N]$. Since $\operatorname{D}[S_N] \cap \mathfrak{N}_\lambda = \operatorname{R}[S_F] \cap \mathfrak{N}_\lambda$, then $\gamma(\lambda)e \in \operatorname{R}[S_F]$. Since $\operatorname{R}[S_F] = \operatorname{ran}(S_F) = \operatorname{ran}(S_F^*)$, from Lemma 3.1 and (2.1) we have:

$$\lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\beta)}} z\gamma(z)e = \lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\beta)}} \left(z\gamma(\lambda)e + (z - \lambda)z(S_F^* - zI)^{-1}\gamma(\lambda)e \right) = 0. \quad \square$$

Theorem 3.3. *Let S be a densely defined closed sectorial operator, $\gamma(z)$ its γ -field, corresponding to the boundary pair $\{\mathcal{H}, \Gamma\}$ of S . Define a set in \mathcal{H} :*

$$\mathcal{D}_0 := \left\{ e \in \mathcal{H} : \lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\alpha)}} |(\mathcal{Q}(z)e, e)_{\mathcal{H}}| < \infty \right\}. \quad (3.3)$$

Then

$$\gamma(\mu)\mathcal{D}_0 = \mathfrak{N}_\mu \cap \operatorname{D}[S_N].$$

for all $\mu \in \mathbb{C} \setminus \Theta(\alpha)$ and

$$\mathcal{D}_0 = \Gamma \operatorname{D}[S_N].$$

Moreover, the following limits exist

$$\begin{aligned} \Omega_0[e, g] &:= - \lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\beta)}} (\mathcal{Q}(z)e, g), \quad e, g \in \mathcal{D}_0, \\ X_0e &:= - \lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\beta)}} \mathcal{G}^*(\bar{z})e, \quad e \in \mathcal{D}_0, \quad \beta \in (\alpha, \pi/2), \end{aligned}$$

and

$$\begin{aligned}\Omega_0[e, g] &= i(\gamma(i)e, \gamma(i)g) + \left((I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i)e, \hat{S}_{FR}^{-1/2} \gamma(i)g \right) = \\ &= i(\gamma(i)e, \gamma(i)g) + S_F^{*-1} [\gamma(i)e, \gamma(i)g], \quad e, g \in \mathcal{D}_0, \\ X_0 e &= i(I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i)e, \quad e \in \mathcal{D}_0.\end{aligned}$$

Proof. Let $e \in \mathcal{H}$. Then using (2.1) and (2.5) we have for $z \in \mathbb{C} \setminus \Theta(\alpha)$

$$\begin{aligned}(\mathcal{Q}(z)e, e)_{\mathcal{H}} &= (z - i)(\gamma(z)e, \gamma(i)e) \\ &= (z - i)(\gamma(i)e + (z - i)((S_F^* - zI)^{-1} \gamma(i)e, \gamma(i)e))\end{aligned}$$

Hence

$$((S_F^* - zI)^{-1} \gamma(i)e, \gamma(i)e) = -\frac{1}{z - i}(\gamma(i)e, \gamma(i)e) + \frac{1}{(z - i)^2}(\mathcal{Q}(z)e, e)_{\mathcal{H}}.$$

The latter equality and (1.2) yields

$$\begin{aligned}\lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\alpha)}} |(\mathcal{Q}(z)e, e)_{\mathcal{H}}| &< \infty \\ \iff \lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\alpha)}} |((S_F^* - zI)^{-1} \gamma(i)e, \gamma(i)e)| &< \infty \\ \iff \gamma(i)e &\in R[S_F] \cap \mathfrak{N}_i.\end{aligned}$$

Let \mathcal{D}_0 be defined by (3.3). Then, using (1.8), (1.16), and Corollary 3.2, one obtains

$$e \in \mathcal{D}_0 \iff \gamma(i)e \in \mathfrak{N}_i \cap D[S_N].$$

Hence $\gamma(\mu)\mathcal{D}_0 = \mathfrak{N}_\mu \cap D[S_N]$ for all $\mu \in \mathbb{C} \setminus \Theta(\alpha)$. Observe that \mathcal{D}_0 is a linear manifold. Equality (1.9) yields that $\Gamma D[S_N] = \mathcal{D}_0$.

Notice that the equality

$$\gamma(z) = \gamma(i) + (z - i)(S_F^* - zI)^{-1} \gamma(i),$$

the inclusion $\gamma(i)\mathcal{D}_0 \subseteq \overline{\text{ran}(S_F^*)}$, and applying Proposition 3.2 leads to

$$\lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus \Theta(\beta)}} z\gamma(z)e = 0, \quad e \in \mathcal{D}_0$$

for $\beta \in (\alpha, \pi/2)$. Applying equality (1.3), we get the rest equalities in Theorem. \square

Clearly the form $\Omega_0[e, g]$ can also be rewritten as follows:

$$\Omega_0[e, g] = i(\gamma(i)e, \gamma(i)g) - i \left(X_0 e, \hat{S}_{FR}^{-1/2} \gamma(i)g \right), \quad e, g \in \mathcal{D}_0.$$

Using expressions for Ω_0 and X_0 , by straightforward calculations one can deduce that

$$\text{Re } \Omega_0[e] = \|(I + iG_F)^{-1} S_{FR}^{-1/2} \gamma(i)e\|^2 = \|X_0 e\|^2, \quad e \in \mathcal{D}_0. \quad (3.4)$$

It follows that the sesquilinear form $\Omega_0[e, g]$ is accretive, and, moreover, the form $\text{Re } \Omega_0$ is closed in the Hilbert space \mathcal{H} . Observe that the form

$$\mathfrak{t}_0[e, g] := \Omega_0[e, g] - i(\gamma(i)e, \gamma(i)g) = \left((I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i)e, \hat{S}_{FR}^{-1/2} \gamma(i)g \right)$$

$$= S_F^{*-1}[\gamma(i)e, \gamma(i)g], \quad e, g \in \mathcal{D}_0,$$

is closed and sectorial in \mathcal{H} . Let the linear relation \mathfrak{T}_0 be associated with \mathfrak{t}_0 by the First Representation Theorem (see [39] for nondensely defined closed sectorial forms). Then define

$$\mathbf{Z}_0 = \mathfrak{T}_0 + iP_{\overline{\mathcal{D}_0}}\gamma^*(i)\gamma(i),$$

where $P_{\overline{\mathcal{D}_0}}$ is the orthogonal projection in \mathcal{H} onto the subspace $\overline{\mathcal{D}_0}$. The linear relation \mathbf{Z}_0 is m -accretive and associated with the form Ω_0 in the sense

$$(\mathbf{Z}_0 e, g)_{\mathcal{H}} = \Omega_0[e, g] \quad \text{for all } e \in \text{dom}(\mathbf{Z}_0) \quad \text{and all } g \in \mathcal{D}_0.$$

Theorem 3.4. *Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair of S . Then the pair $\langle \mathbf{Z}_0, X_0 \rangle$ corresponds to the Kreĭn-von Neumann extension S_N of the operator S in accordance with Theorem 2.4.*

Proof. It follows from (2.3) and from Theorem 3.3 that

$$S_N[u, v] = l[u, v] + \Omega_0[\Gamma u, \Gamma v] + 2(X_0 \Gamma u, S_{FR}^{1/2} \mathcal{P}_F v), \quad u, v \in \text{D}[S_N]. \quad (3.5)$$

Let the pair $\langle Z_N, X_N \rangle$ corresponds to S_N in accordance with Theorem 2.4, $\text{dom}(Z_N) = \text{dom}(S_N)$, $\text{dom}(X_N) = \Gamma \text{dom}(S_N)$. Then

$$(S_N u, v) = l[u, v] + (Z_N u, \Gamma v)_{\mathcal{H}} + 2(X_N \Gamma u, S_{FR}^{1/2} \mathcal{P}_F v), \quad u \in \text{dom}(S_N), \quad v \in \mathfrak{L}. \quad (3.6)$$

Then (3.5) and (3.6) imply for $v \in \text{D}[S]$ that

$$(X_0 \Gamma u, S_{FR}^{1/2} v) = (X_N \Gamma u, S_{FR}^{1/2} v).$$

Hence $X_N = X_0 \upharpoonright \Gamma \text{dom}(S_N)$. Further

$$\Omega_0[\Gamma u, \Gamma v] = (Z_N u, \Gamma v)_{\mathcal{H}}, \quad u \in \text{dom}(S_N), \quad v \in \text{D}[S_N].$$

Therefore, m -accretive linear relation

$$\mathbf{Z}_N = \{ \{ \Gamma u, Z_N u \}, \quad u \in \text{dom}(S_N) \}$$

is associated with the form Ω_0 . It follows the equality

$$\mathbf{Z}_N = \mathbf{Z}_0. \quad \square$$

Remark 3.5. If the set \mathcal{D}_0 in Theorem 3.3 is trivial, then the operator S admits a unique m -sectorial extension, namely the Friedrichs extension S_F .

Let

$$S_N[u, v] = \left((I + iG_N) S_{NR}^{1/2} u, S_{NR}^{1/2} v \right), \quad u, v \in \text{D}[S_N].$$

Since $S_N[u, v] = S_F[u, v]$, for all $u, v \in \text{D}[S]$, there exists an isometry U_F mapping $\overline{\text{ran}(S_F)}$ onto $\overline{\text{ran}(S_N)}$ such that (see [7, 8])

$$S_{NR}^{1/2} u = U_F S_{FR}^{1/2} u, \quad u \in \text{D}[S],$$

$$G_N U_F = U_F G_F,$$

$$S_{NR}^{1/2} \varphi_\mu = \mu U_F (I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \varphi_\mu, \quad \varphi_\mu \in \mathfrak{N}_\mu \cap \text{D}[S_N].$$

It follows that

$$S_{NR}^{1/2} u = U_F S_{FR}^{1/2} \mathcal{P}_F u + U_F X_0 \Gamma u, \quad (3.7)$$

Description of all closed sesquilinear forms associated with m -sectorial extensions of operator S in the terms of boundary pair has been obtained in [7].

Definition 3.6 ([7]). *A pair $\{\mathcal{H}', \Gamma'\}$ is called boundary pair of the operator S , if \mathcal{H}' is a Hilbert space, and $\Gamma' : D[S_N] \rightarrow \mathcal{H}'$ is a linear operator such that $\ker(\Gamma') = D[S]$, $\text{ran}(\Gamma') = \mathcal{H}'$.*

Since $D[S]$ is a subspace in $D[S_N]$, the boundary pairs $\{\mathcal{H}', \Gamma'\}$ for operator S exist.

Theorem 3.7 ([7, 11]). *Let $\{\mathcal{H}', \Gamma'\}$ be a boundary pair of the operator S in the sense of Definition 3.6. Then the formula*

$$\begin{aligned} \tilde{S}[u, v] &= S_N[u, v] + \omega'[\Gamma'u, \Gamma'v] + 2(X'\Gamma'u, S_{NR}^{1/2}v), \\ u, v &\in D[\tilde{S}] = \Gamma'^{-1} D[\omega'] \end{aligned} \quad (3.8)$$

establish a bijective correspondence between all closed forms associated with m -sectorial extensions \tilde{S} of S and all pairs $\langle \omega', X' \rangle$, where

- 1) ω' is a closed and sectorial sesquilinear in the Hilbert space \mathcal{H}' ;
- 2) $X' : \text{dom}(\omega') \rightarrow \overline{\text{ran}(S)}$ is a linear operator, such that for some $\delta \in [0, 1)$:

$$\|X'e\|^2 \leq \delta^2 \text{Re } \omega'[e],$$

for all $e \in \text{dom}(\omega')$.

Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair of the operator S in the sense of Definition 2.1. Set

$$\begin{aligned} \mathcal{H}' &= \mathcal{D}_0 (= \text{dom}(\Omega_0)), \\ (e, g)_{\mathcal{H}'} &= (e, g)_{\mathcal{H}} + \text{Re } \Omega_0[e, g] = (e, g)_{\mathcal{H}} + (X_0e, X_0g), \\ \Gamma' &= \Gamma \upharpoonright D[S_N] = \Gamma \upharpoonright (D[S] \dot{+} \gamma(i)\mathcal{D}_0). \end{aligned} \quad (3.9)$$

Then \mathcal{H}' is a Hilbert space w.r.t. the inner product $(\cdot, \cdot)_{\mathcal{H}'}$ and $\{\mathcal{H}', \Gamma'\}$ is boundary pair of the operator S in the sense of Definition 3.6. Note that

- 1) the operators X_0 and $\gamma(\lambda)$ are continuous from \mathcal{H}' into \mathfrak{H} ,
- 2) the sesquilinear form Ω_0 is continuous in \mathcal{H}' .

Further, using Theorem 2.4 and representation (3.5) for the form $S_N[u, v]$, we are going to established additional conditions on the pairs $\langle \mathbf{Z}, X \rangle$ that determine m -sectorial extensions of the operator S in accordance with Theorem 2.5.

Theorem 3.8. *Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair of S . Then the pair $\langle \mathbf{Z}, X \rangle$ determines an m -sectorial extension \tilde{S} of S , see Theorem 2.5 and Remark 2.6, if and only if the following conditions are fulfilled:*

- 1) $\text{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0$;
- 2) the sesquilinear form

$$\begin{aligned} \omega[e, g] &= (\mathbf{Z}e, g)_{\mathcal{H}} - \Omega_0[e, g] - 2((X - X_0)e, X_0g) \\ &= (\mathbf{Z}e, g)_{\mathcal{H}} + \Omega_0^*[e, g] - 2(Xe, X_0g), \\ e, g &\in \text{dom}(\tilde{S}) = \Gamma \text{dom}(\tilde{S}) \end{aligned} \quad (3.10)$$

is sectorial and admits a closure in the Hilbert space \mathcal{H}' ;

3) $\|(X - X_0)e\|^2 \leq \delta^2 \operatorname{Re} \omega[e]$, $e \in \operatorname{dom}(\mathbf{Z})$ for some $\delta \in [0, 1)$.

Moreover, the closed sesquilinear form associated with \tilde{S} is given by

$$\begin{aligned} \tilde{S}[u, v] &= l[u, v] + \mathbf{Z}[\Gamma u, \Gamma v] + 2(\overline{X}\Gamma u, S_{FR}^{1/2}\mathcal{P}_F v), \\ u, v &\in \operatorname{D}[\tilde{S}] = \Gamma^{-1} \operatorname{dom}(\overline{\omega}), \end{aligned} \quad (3.11)$$

where \overline{X} is continuous extension of X on the domain $\operatorname{dom}(\overline{\omega})$ of the closure $\overline{\omega}$ of ω and

$$\mathbf{Z}[e, g] := \overline{\omega}[e, g] - \Omega_0^*[e, g] + 2(\overline{X}e, X_0g), \quad e, g \in \operatorname{dom}(\overline{\omega}). \quad (3.12)$$

Proof. Let \tilde{S} be an m -sectorial extension of S determined by the pair $\langle \mathbf{Z}, X \rangle$ in accordance with Theorem 2.4. Note, that since \tilde{S} is m -sectorial extension of S , we have (see (1.10)) $\operatorname{dom}(\tilde{S}) \subset \operatorname{D}[\tilde{S}] \subseteq \operatorname{D}[S_N]$, and $\Gamma \operatorname{dom}(\tilde{S})$ is a core of the linear manifold $\Gamma \operatorname{D}[\tilde{S}]$. Then

$$(\tilde{S}u, v) = l[u, v] + (\mathbf{Z}\Gamma u, \Gamma v)_{\mathcal{H}} + 2(X\Gamma u, S_{FR}^{1/2}\mathcal{P}_F v), \quad u, v \in \operatorname{dom}(\tilde{S}).$$

Using (3.5), one obtains:

$$\begin{aligned} (\tilde{S}u, v) &= S_N[u, v] + (\mathbf{Z}\Gamma u, \Gamma v)_{\mathcal{H}} - \Omega_0[\Gamma u, \Gamma v] \\ &\quad + 2((X - X_0)\Gamma u, S_{FR}^{1/2}\mathcal{P}_F v), \quad u, v \in \operatorname{dom}(\tilde{S}). \end{aligned}$$

From (3.7) $S_{FR}^{1/2}\mathcal{P}_F v = U_F^* S_{NR}^{1/2} v - X_0 \Gamma v$. Hence,

$$\begin{aligned} (\tilde{S}u, v) &= S_N[u, v] + (\mathbf{Z}\Gamma u, \Gamma v)_{\mathcal{H}} - \Omega_0[\Gamma u, \Gamma v] \\ &\quad - 2((X - X_0)\Gamma u, X_0 \Gamma v) + 2(U_F(X - X_0)\Gamma u, S_{NR}^{1/2} v) \\ &= S_N[u, v] + \omega[\Gamma u, \Gamma v] + 2(U_F(X - X_0)\Gamma u, S_{NR}^{1/2} v) \\ &= S_N[u, v] + \omega[\Gamma' u, \Gamma' v] + 2(\tilde{X}\Gamma' u, S_{NR}^{1/2} v), \quad u, v \in \operatorname{dom}(\tilde{S}), \end{aligned}$$

where ω is given by (3.10) and $\tilde{X} = U_F(X - X_0)$. From Theorem 3.7 it follows that ω is sectorial form, $\operatorname{dom}(\omega) = \operatorname{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0 = \mathcal{H}'$ and

$$\|\tilde{X}e\|^2 = \|(X - X_0)e\|^2 \leq \delta^2 \operatorname{Re} \omega[e]$$

for all $e \in \operatorname{dom}(\mathbf{Z})$, where $\delta \in [0, 1)$. Moreover, the form ω admits closure $\overline{\omega}$ in the Hilbert space \mathcal{H}' , and \tilde{X} can be extended on $\operatorname{dom}(\overline{\omega})$ by continuity as a linear operator from $\operatorname{dom}(\overline{\omega})$ with the inner product

$$(e, g)_{\overline{\omega}} = (e, g)_{\mathcal{H}'} + \operatorname{Re} \overline{\omega}[e, g].$$

Since X_0 is continuous from \mathcal{H}' into \mathfrak{H} , the operator X admits a continuation \overline{X} on $\operatorname{dom}(\overline{\omega})$. It follows that the form \mathbf{Z} given by (3.12) is well defined and the closed form $\tilde{S}[u, v]$ associated with \tilde{S} is of the form (3.11).

Conversely, let conditions (1)–(3) of the theorem be fulfilled. Denote by $\overline{\omega}$ the closure in the Hilbert space \mathcal{H}' of the sesquilinear form ω given by (3.10), and by $\overline{X'}$ the continuation of the operator $\tilde{X} = U_F(X - X_0)$ on $\operatorname{dom}(\overline{\omega})$, which exists due condition (2). Then, by Theorem 3.7, the pair $\langle \overline{\omega}, \overline{X'} \rangle$ determines by (3.8) a closed

sectorial form $\tilde{S}[u, v]$ associated with some m -sectorial extension \tilde{S} of the operator S . \square

Remark 3.9. We can rewrite condition (3) of Theorem 3.8 in slightly different form. Let us find the real part of the form $\omega[e, e]$. We have:

$$\omega[e, e] = (\mathbf{Z}e, e)_{\mathcal{H}} - \Omega_0[e, e] - 2((X - X_0)e, X_0e).$$

Using (3.4), we obtain:

$$\begin{aligned} \operatorname{Re} \omega[e, e] &= \operatorname{Re} (\mathbf{Z}e, e)_{\mathcal{H}} - \|X_0e\|^2 + 2\|X_0e\|^2 - 2\operatorname{Re} (Xe, X_0e) = \\ &= \operatorname{Re} (\mathbf{Z}e, e)_{\mathcal{H}} + \|X_0e\|^2 - 2\operatorname{Re} (Xe, X_0e) = \\ &= \operatorname{Re} (\mathbf{Z}e, e)_{\mathcal{H}} + \|(X - X_0)e\|^2 - \|Xe\|^2. \end{aligned}$$

Then the inequalities

$$\|(X - X_0)e\|^2 \leq \delta^2 \operatorname{Re} \omega[e] = \delta^2 (\operatorname{Re} (\mathbf{Z}e, e)_{\mathcal{H}} + \|(X - X_0)e\|^2 - \|Xe\|^2)$$

and $0 \leq \delta < 1$ imply

$$M\|(X - X_0)e\|^2 \leq \operatorname{Re} (\mathbf{Z}e, e)_{\mathcal{H}} - \|Xe\|^2,$$

where $M = \frac{1 - \delta^2}{\delta^2} > 0$.

Thus, condition 3 can be rewritten as

$$\operatorname{Re} (\mathbf{Z}e, e)_{\mathcal{H}} - \|Xe\|^2 \geq M\|(X - X_0)e\|^2, \quad M > 0.$$

4. NONNEGATIVE SYMMETRIC OPERATOR AND ITS QUASI-SELFADJOINT m -ACCRETIVE EXTENSIONS

In this section we will consider a densely defined closed nonnegative symmetric operator A and parameterize all its quasi-selfadjoint m -accretive extensions in terms of abstract boundary conditions. We will use a boundary pair and boundary triplets defined in Definitions 2.1, 2.2, and 2.3. In this case if $\{\mathcal{H}, \Gamma\}$ is the boundary pair for A in the sense of Definition 2.1, then the sesquilinear form Ω_0 and the linear operator X_0 defined on the linear manifold $\mathcal{D}_0 = \Gamma D[A_N]$ (see Theorem 3.3) are of the form

$$\begin{aligned} \Omega_0[e, g] &= i(\gamma(i)e, \gamma(i)g) + \left(\hat{A}_F^{-1/2} \gamma(i)e, \hat{A}_F^{-1/2} \gamma(i)g \right) \\ X_0e &= i\hat{A}_F^{-1/2} \gamma(i)e, \quad e, g \in \mathcal{D}_0. \end{aligned}$$

In addition, from (1.18) it follows that

$$\begin{aligned} A_N[u, v] &= \left(\left(A_F^{1/2} \mathcal{P}_{z,F} u + z \hat{A}_F^{-1/2} \mathcal{P}_z u \right), \left(A_F^{1/2} \mathcal{P}_{z,F} v + z \hat{A}_F^{-1/2} \mathcal{P}_z v \right) \right) \\ &= \left(\left(A_F^{1/2} (u - \gamma(z) \Gamma u) + z \hat{A}_F^{-1/2} \gamma(z) \Gamma u \right), \left(A_F^{1/2} (v - \gamma(z) \Gamma v) + z \hat{A}_F^{-1/2} \gamma(z) \Gamma v \right) \right), \\ u, v &\in D[A_N] = D[A_F] \dot{+} (\mathfrak{N}_z \cap \operatorname{ran}(A_F^{1/2})) = D[A_F] \dot{+} \gamma(z) \mathcal{D}_0. \end{aligned} \quad (4.1)$$

It is established in [6] (see also [14]) that the following assertions are equivalent for m -accretive extension \tilde{A} of A :

- (i) A is quasi-selfadjoint extension;
- (ii) $\text{dom}(\tilde{A}) \subseteq D[A_N]$ and $\text{Re}(\tilde{A}f, f) \geq A_N[f]$ for all $f \in \text{dom}(\tilde{A})$.

Observe that the operator L defined in (2.6) is of the form

$$\text{dom}(L) = \text{dom}(A^*), \quad Lu = A^*u - 2iu_i,$$

where $u = u_F + u_i$, $u_F \in \text{dom}(A_F)$, $u_i \in \mathfrak{N}_i$. If $\{\mathcal{H}, \Gamma\}$ is a boundary pair for A (see Definition 2.1), then

$$Lu = A^*u - 2i\gamma(i)\Gamma u, \quad u \in \text{dom}(A^*).$$

Proposition 4.1. *Let A be a closed densely defined nonnegative symmetric operator in \mathfrak{H} and let $\{\mathcal{H}, \Gamma\}$ be its boundary pair (in the sense of Definition 2.1). Assume $\mathcal{D}_0 \neq \{0\}$. Then a pair $\langle \mathbf{Z}, X \rangle$ determines a quasi-selfadjoint m -accretive extension \tilde{A} of A in accordance with Theorem 2.5 if and only if the following conditions hold true*

- 1) $\text{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0$,
- 2) $X = X_0 \upharpoonright \text{dom}(\mathbf{Z}) = i\hat{A}_F^{-1/2}\gamma(i) \upharpoonright \text{dom}(\mathbf{Z})$.

Proof. Let \tilde{A} be a quasi-selfadjoint m -accretive extension of the operator A . Then $\text{dom}(\tilde{A}) \subseteq D[A_N]$. By Theorem 2.4 this implies the inclusion $\text{dom}(\mathbf{Z}) \subseteq \Gamma D[A_N] = \mathcal{D}_0$. Taking into account the decomposition $\text{dom}(A^*) = \text{dom}(A_F) \dot{+} \mathfrak{N}_i$, from (2.18) for $\text{dom}(\tilde{A}) \ni u = u_F + u_i$, $u_F \in \text{dom}(A_F)$, $u_i \in \mathfrak{N}_i$ we have

$$\begin{aligned} X\Gamma u &= Mu = \frac{1}{2} \left(\hat{A}_{FR}^{-1/2}(\tilde{A}u + i\mathcal{P}_i u) - (I + iG_F)A_F^{1/2}\mathcal{P}_F u \right) = \\ &= \frac{1}{2} \left(\hat{A}_F^{-1/2}(A^*u + iu_i) - A_F^{1/2}u_F \right) = \frac{1}{2} \left(\hat{A}_F^{-1/2}(A_F u_F + 2iu_i) - A_F^{1/2}u_F \right) = \\ &= i\hat{A}_F^{-1/2}\gamma(i)\Gamma u = X_0\Gamma u. \end{aligned}$$

Now consider a pair $\langle \mathbf{Z}, X \rangle$, where \mathbf{Z} is m -accretive linear relation in \mathcal{H} such that (a) $\text{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0$ and (b) $\text{Re}(\mathbf{Z}e, e)_{\mathcal{H}} \geq \|X_0 e\|^2$ for all $e \in \text{dom}(\mathbf{Z})$. This pair determines an m -accretive extension \tilde{A} . Let us prove that $\tilde{A} \subseteq A^*$. Note that for all $u \in \mathfrak{L}$, $v \in \mathfrak{H}$

$$\begin{aligned} (\Phi(\lambda)X_0\Gamma u, v) &= i \left(\hat{A}_F^{-1/2}\gamma(i)\Gamma u, A_F^{1/2}(A_F - \bar{\lambda}I)^{-1}v \right) = \\ &= i \left((A_F - \lambda I)^{-1}\gamma(i)\Gamma u, v \right). \end{aligned}$$

So,

$$\Phi(\lambda)X_0\Gamma u = i(A_F - \lambda I)^{-1}\gamma(i)\Gamma u \subset \text{dom}(A_F). \quad (4.2)$$

Using (4.2) one gets

$$\begin{aligned} q(\lambda) - 2\Phi(\lambda)X_0 &= \\ &= \gamma(i) + (\lambda + i)(A_F - \lambda I)^{-1}\gamma(i) - 2i(A_F - \lambda I)^{-1}\gamma(i) = \\ &= \gamma(i) + (\lambda - i)(A_F - \lambda I)^{-1}\gamma(i) = \gamma(\lambda). \end{aligned} \quad (4.3)$$

From boundary conditions (2.10) for $u \in \mathfrak{L}$ we have:

$$u \in \text{dom}(\tilde{A}) \Rightarrow u - (q(\lambda) - 2\Phi(\lambda)X_0)\Gamma u \in \text{dom}(A_F) \Rightarrow u - \gamma(\lambda)\Gamma u \in \text{dom}(A_F),$$

and, therefore, $u \in \text{dom}(A_F) \dot{+} \mathfrak{N}_\lambda = \text{dom}(A^*)$. Further, for $u = \mathcal{P}_{\lambda,F}u + \mathcal{P}_\lambda u$

$$\begin{aligned} \tilde{A}u &= A_F(u - (q(\lambda) - 2\Phi(\lambda)X_0)\Gamma u) + \lambda(q(\lambda) - 2\Phi(\lambda)X_0)\Gamma u = \\ &= A_F(u - \gamma(\lambda)\Gamma u) + \lambda\gamma(\lambda)\Gamma u = \\ &= A_F\mathcal{P}_{\lambda,F}u + \lambda\mathcal{P}_\lambda u = A^*(\mathcal{P}_{\lambda,F}u + \mathcal{P}_\lambda u). \end{aligned}$$

So, $\tilde{A} \subseteq A^*$. □

Theorem 4.2. *Let $\{\mathcal{H}, \Gamma\}$ and $\{\mathcal{H}, G_*, \Gamma\}$ be a boundary pair for A and the corresponding boundary triplet for L , see Definition 2.3. Assume $\mathcal{D}_0 \neq \{0\}$. Then there is a bijective correspondence between all m -accretive quasi-selfadjoint extensions \tilde{A} of A and all m -accretive linear relations \mathbf{Z} in \mathcal{H} such that $\text{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0$ and:*

$$\text{Re}(\mathbf{Z}e, e) \geq \|\hat{A}_F^{-1/2}\gamma(i)e\|^2, \quad \forall e \in \text{dom}(\mathbf{Z}).$$

This correspondence is given by

$$\begin{aligned} \text{dom}(\tilde{A}) &= \{u \in \text{dom}(A^*) : G_*u \in (\mathbf{Z} - 2i\gamma^*(i)\gamma(i))\Gamma u\}, \\ \tilde{A}u &= A^*u. \end{aligned} \tag{4.4}$$

Moreover,

1) a number $\lambda \in \rho(A_F)$ is a regular point of \tilde{A} if and only if

$$\left(\mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda)\right)^{-1} \in \mathbf{L}(\mathcal{H}),$$

and,

$$(\tilde{A} - \lambda I)^{-1} = (A_F - \lambda I)^{-1} + \gamma(\lambda) \left(\mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda)\right)^{-1} \gamma^*(\bar{\lambda}); \tag{4.5}$$

2) a number $\lambda \in \rho(A_F)$ is an eigenvalue of \tilde{A} if and only if

$$\ker \left(\mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda)\right) \neq \{0\},$$

and,

$$\ker(\tilde{A} - \lambda I) = \gamma(\lambda) \ker \left(\mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda)\right).$$

Proof. We will use (2.10). Due to (4.3) the boundary condition 1) in (2.10) is fulfilled. Let us transform boundary condition 2). Due to (2.7) we have for $\lambda \in \rho(A_F)$

$$G_*(f + q(\lambda)e) = \gamma^*(\bar{\lambda})(A_F - \lambda I)f + \mathcal{Q}^*(\bar{\lambda})e, \quad f \in \text{dom}(A_F), \quad e \in \text{dom}(A_F).$$

So, we have

$$G_*(u + 2\Phi(\lambda)X\Gamma u) = G_*(u + (q(\lambda) - \gamma(\lambda))\Gamma u) =$$

$$\begin{aligned}
&= G_*(u + 2i(A_F - \lambda I)^{-1}\gamma(i)\Gamma u) = \\
&= G_*u + 2i\gamma^*(\bar{\lambda})\gamma(i)\Gamma u = \\
&= \gamma^*(\bar{\lambda})(A_F - \lambda I)\mathcal{P}_{\lambda, F}u + \mathcal{Q}^*(\bar{\lambda})\Gamma u.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbf{W}(\lambda) &= \mathbf{Z} - \mathcal{Q}^*(\bar{\lambda}) + 2\mathcal{G}(\lambda)X_0 \\
&= \mathbf{Z} - (\lambda + i)\gamma^*(\bar{\lambda})\gamma(i) + 2(\lambda + i)\gamma^*(i)\Phi(\lambda)X_0 \\
&= \mathbf{Z} - (\lambda + i)(\gamma^*(i) + (\lambda + i)\gamma^*(i)(A_F - \lambda I)^{-1})\gamma(i) - 2(\lambda + i)i\gamma^*(i)(A_F - \lambda I)^{-1}\gamma(i) \\
&= \mathbf{Z} - (\lambda + i)\gamma^*(i)(I + (\lambda + i)(A_F - \lambda I)^{-1} - 2i(A_F - \lambda I)^{-1})\gamma(i) \\
&= \mathbf{Z} - (\lambda + i)\gamma^*(i)\gamma(\lambda) = \mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda).
\end{aligned}$$

Then

$$\mathbf{Z} + 2\mathcal{G}(\lambda)X_0 = \mathbf{Z} + \mathcal{Q}^*(\bar{\lambda}) - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda).$$

So, for the boundary condition 2) from (2.10) one has

$$\begin{aligned}
G_*u + 2i\gamma^*(\bar{\lambda})\gamma(i)\Gamma u &\in \left(\mathbf{Z} + \mathcal{Q}^*(\bar{\lambda}) - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda) \right) \Gamma u \\
&\iff G_*u \in \left(\mathbf{Z} + \mathcal{Q}^*(\bar{\lambda}) - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda) - 2i\gamma^*(\bar{\lambda})\gamma(i) \right) \Gamma u \\
&\iff G_*u \in \left(\mathbf{Z} + (\lambda + i)\gamma^*(\bar{\lambda})\gamma(i) - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda) - 2i\gamma^*(\bar{\lambda})\gamma(i) \right) \Gamma u \\
&\iff G_*u \in \left(\mathbf{Z} + \frac{\lambda - i}{\lambda + i}\mathcal{Q}^*(\bar{\lambda}) - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda) \right) \Gamma u.
\end{aligned}$$

Further, using that $\mathcal{Q}(\lambda) = (\lambda - i)\gamma^*(i)\gamma(i)$, we get

$$\begin{aligned}
&(\mathbf{Z} + (\lambda - i)\gamma^*(\bar{\lambda})\gamma(i) - (\lambda + i)\gamma^*(i)\gamma(\lambda)) \Gamma u \\
&= (\mathbf{Z} + (\lambda - i)(\gamma^*(i) + (\lambda + i)\gamma^*(i)(A_F - \lambda I)^{-1})\gamma(i) \\
&\quad - (\lambda + i)\gamma^*(i)(\gamma(i) + (\lambda - i)(A_F - \lambda I)^{-1}\gamma(i))) \Gamma u \\
&= \left(\mathbf{Z} + \gamma^*(i)((\lambda - i)I + (\lambda^2 + 1)(A_F - \lambda I)^{-1}) \right. \\
&\quad \left. - ((\lambda + i)I + (\lambda^2 + 1)(A_F - \lambda I)^{-1}) \right) \gamma(i)\Gamma u \\
&= (\mathbf{Z} - 2i\gamma^*(i)\gamma(i))\Gamma u.
\end{aligned}$$

□

Remark 4.3. The boundary condition (4.4) also can be written for any $\lambda \in \rho(\tilde{A}) \cap \rho(A_F)$ as

$$\text{dom}(\tilde{A}) = \left\{ u \in \text{dom}(A^*) : \gamma^*(\bar{\lambda})(A_F - \lambda I)(u - \gamma(\lambda)\Gamma u) \in \left(\mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda) \right) \Gamma u \right\},$$

and

$$\tilde{A}u = A^*u = A_F(u - \gamma(\lambda)\Gamma u) + \lambda\gamma(\lambda)\Gamma u.$$

From Theorems 3.8, 4.2 we obtain

Corollary 4.4. *Let \mathbf{Z} be m -accretive linear relation, corresponding to a quasi-selfadjoint m -accretive extension \tilde{A} of A by the Theorem 4.2. Then extension \tilde{A} is a sectorial (nonnegative) if and only if*

- 1) $\text{dom}(\mathbf{Z}) \subseteq \mathcal{H}' (= \text{dom}(\Omega_0) = \mathcal{D}_0)$;
- 2) *the form $\tilde{\omega}[e, g] = (\mathbf{Z}e, g)_{\mathcal{H}} - \Omega_0[e, g]$ is sectorial (nonnegative).*

Remark 4.5. The form $\tilde{\omega}$ admits a closure in the Hilbert space \mathcal{H}' defined by (3.9). Actually, since $\tilde{\omega}[e, g] = (\mathbf{Z}e, g)_{\mathcal{H}} - \Omega_0[e, g]$ is sectorial, the form

$$\eta[e, f] = (\mathbf{Z}e, g)_{\mathcal{H}} - i(\gamma(i)e, \gamma(i)f), \quad e, f \in \mathcal{H}' (= \mathcal{D}_0)$$

is sectorial as well. If

$$\begin{aligned} \lim_{n \rightarrow \infty} e_n &= 0 \quad \text{in } \mathcal{H}', \\ \lim_{m, n \rightarrow \infty} \tilde{\omega}[e_n - e_m] &= 0, \end{aligned}$$

then

$$\begin{aligned} \lim_{n \rightarrow \infty} e_n &= 0 \quad \text{in } \mathcal{H}, \quad \lim_{n \rightarrow \infty} \text{Re } \Omega[e_n] = \lim_{n \rightarrow \infty} \|X_0 e_n\|^2 = 0, \\ \lim_{n \rightarrow \infty} \gamma(i)e_n &= 0 \quad \text{in } \mathfrak{H}. \end{aligned}$$

Since linear relation \mathbf{Z} is m -accretive and $\mathbf{Z} - i\gamma^*(i)\gamma(i)$ is sectorial, we get $\lim_{n \rightarrow \infty} (\mathbf{Z}e_n, e_n)_{\mathcal{H}} = 0$ (see [28]).

Next we will find relationships between

- a boundary triplet $\{\mathcal{H}, \Gamma_1, \Gamma_0\}$ for A^* given by Definition 1.2 and boundary triplets $\{\mathcal{H}, G, \Gamma\}$, $\{\mathcal{H}, G_*, \Gamma\}$ of Definitions 2.2 and 2.3;
- parameterizations of quasi-selfadjoint m -accretive extensions given by Theorem 1.3 and Theorem 4.2.

Let $\{\mathcal{H}, \Gamma_1, \Gamma_0\}$ be a boundary triplet of A^* (see Definition 1.2) such that $\ker(\Gamma_0) = \text{dom}(A_F)$. Then

- 1) since $\text{dom}(A_F)$ is a core of $D[A]$ and $\ker(\Gamma_0) = \text{dom}(A_F)$, we can define a boundary pair $\{\mathcal{H}, \bar{\Gamma}_0\}$ where $\bar{\Gamma}_0$ is a continuation of Γ_0 onto $\mathfrak{L} = D[A] \dot{+} \mathfrak{N}_i$ from $\text{dom}(A^*) = \text{dom}(A_F) \dot{+} \mathfrak{N}_i$;
- 2) it follows that

$$\gamma(\lambda) = (\bar{\Gamma}_0 \upharpoonright \mathfrak{N}_\lambda)^{-1} = \Gamma_0(\lambda);$$

- 3) because relation (1.23) can be rewritten as

$$M_0(\lambda) - M_0(z) = (\lambda - z)\gamma^*(\bar{z})\gamma(\lambda),$$

using (2.5), one gets

$$\mathcal{Q}(\lambda) = (\lambda - i)\gamma^*(i)\gamma(\lambda) = \frac{\lambda - i}{\lambda + i}(M_0(\lambda) - M_0(-i));$$

so,

$$M_0(\lambda) - M_0(-i) = \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda); \tag{4.6}$$

- 4) equation (4.6) yields that the linear manifolds \mathcal{D}_0 in Theorems 1.3 and Theorem 3.3 coincide and

$$\tau[h, g] = (M_0(-i)h, g)_{\mathcal{H}} + \Omega_0[h, g], \quad h, g \in \mathcal{D}_0;$$

- 5) comparing resolvent formulas (1.24) and (4.5) we get that the linear relation \mathbf{Z} from Theorem 4.2 and the linear relation $\tilde{\mathbf{T}}$ from Theorem 1.3 (see (1.20), (1.25)) are connected by the equality

$$\mathbf{Z} = \tilde{\mathbf{T}} - M(-i). \quad (4.7)$$

Proposition 4.6. *Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for nonnegative symmetric operator A . Let \tilde{A} be a quasi-selfadjoint m -accretive extension of A and let \mathbf{Z} be the corresponding linear relation in \mathcal{H} (see Theorem 4.2). Then*

$$\mathbf{Z}^* + 2i\gamma^*(i)\gamma(i)$$

corresponds to the adjoint extension \tilde{A}^ .*

Proof. The proof is easy, if we recall that to the adjoint extension \tilde{A}^* corresponds the adjoint linear relation $\tilde{\mathbf{T}}^*$. Since

$$\tilde{\mathbf{T}} = \mathbf{Z} + M(-i).$$

Then

$$\tilde{\mathbf{T}}^* = \mathbf{Z}^* + M^*(-i).$$

Again, it follows from (4.7), equality $M^*(z) = M(\bar{z})$, and (1.23) that the adjoint extension \tilde{A}^* corresponds to

$$\tilde{\mathbf{T}}^* - M(-i) = \mathbf{Z}^* + M^*(-i) - M(-i) = \mathbf{Z}^* + 2i\gamma^*(i)\gamma(i). \quad \square$$

5. m -SECTORIAL EXTENSIONS OF A SYMMETRIC OPERATOR IN THE MODEL OF TWO POINT INTERACTIONS ON A PLANE

Let $y_1, y_2 \in \mathbb{R}^2$. Consider in the Hilbert space $L_2(\mathbb{R}^2)$ the operator A given by:

$$\begin{aligned} \text{dom}(A) &= \{f(x) \in W_2^2(\mathbb{R}^2) : f(y_1) = f(y_2) = 0, \quad k = 1, 2\}, \\ Af &= -\Delta f, \end{aligned} \quad (5.1)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $W_2^2(\mathbb{R}^2)$ is a Sobolev space, and

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

is Laplacian.

The operator A is a densely defined closed nonnegative symmetric with defect indices $(2, 2)$ [2]. Such operators are basic in the models of point interactions [2]. In the case of one point the corresponding operator

$$\text{dom}(A_y) = \{f(x) \in W_2^2(\mathbb{R}^2) : f(y) = 0\}, \quad A_y f = -\Delta f$$

admits a unique nonnegative selfadjoint extension [1, 24], the free Hamiltonian:

$$\text{dom}(A_F) = W_2^2(\mathbb{R}^2), \quad A_F f = -\Delta f,$$

Therefore, A_y has no m -sectorial and quasi-selfadjoint m -accretive extensions. All m -accretive extensions of A_y have been described in [15]. For two and more point interactions the relation $A_F \neq A_N$ holds [1]. In this section we apply Theorems 2.5 and 3.8 for a parametrization of all m -sectorial extensions of the operator A . It is convenient to use the Fourier transform and the momentum representation of A :

$$\hat{A}\hat{f}(p) = |p|^2\hat{f}(p),$$

$$\text{dom}(\hat{A}) = \left\{ \hat{f}(p) \in L_2(\mathbb{R}^2, dp) : \begin{array}{l} 1) |p|^2\hat{f}(p) \in L_2(\mathbb{R}^2, dp), \\ 2) \int_{\mathbb{R}^2} \hat{f}(p)e^{ipy_1} dp = \int_{\mathbb{R}^2} \hat{f}(p)e^{ipy_2} dp = 0. \end{array} \right\}$$

For a one-center point interaction this method has been used in [15]. In this paper we omit details in the momentum representation and present final results in the coordinate representation.

The Friedrichs extension of the operator A is the free Hamiltonian A_F and $A_F^{1/2} = (-\Delta)^{1/2}$ is a pseudodifferential operator of the form:

$$\text{dom}(A_F^{1/2}) = D[A_F] = W_2^1(\mathbb{R}^2),$$

$$A_F^{1/2}f(x) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |p| \exp(i(x-y)p) f(y) dy dp,$$

where $W_2^k(\mathbb{R}^2)$, $k = 1, 2$, are the Sobolev spaces. Note that, see [2], the resolvent is of the form

$$(A_F - \lambda I)^{-1}f(x) = \frac{i}{4} \int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x-y|) f(y) dy, \quad f \in L_2(\mathbb{R}^2),$$

$$\lambda \in \mathbb{C} \setminus [0, +\infty), \quad \text{Im } \sqrt{\lambda} > 0,$$

where $H_0^{(1)}(\cdot)$ denotes the Hankel function of first kind and order zero [36]. It is well known [2] that

$$\mathfrak{N}_\lambda = \left\{ \frac{\pi i}{2} \sum_{k=1}^2 H_0^{(1)}(\sqrt{\lambda}|x-y_k|) c_k, \quad c_1, c_2 \in \mathbb{C} \right\},$$

$$\lambda \in \mathbb{C} \setminus [0, +\infty), \quad \text{Im } \sqrt{\lambda} > 0$$

is the defect subspace of A , corresponding to λ . Therefore, for the linear manifold \mathfrak{L} defined by (1.17) we have

$$\mathfrak{L} = W_2^1(\mathbb{R}^2) \dot{+} \mathfrak{N}_\lambda$$

$$= \left\{ f(x) + \frac{\pi i}{2} \sum_{k=1}^2 H_0^{(1)}(\sqrt{\lambda}|x-y_k|) c_k, \quad f \in W_2^1(\mathbb{R}^2), \quad c_1, c_2 \in \mathbb{C} \right\},$$

where λ is a number from $\mathbb{C} \setminus [0, +\infty)$. Now, let $\mathcal{H} = \mathbb{C}^2$ and set

$$\Gamma \left(f(x) + \frac{\pi i}{2} \sum_{k=1}^2 H_0^{(1)}(\sqrt{\lambda}|x-y_k|) c_k \right) = \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{C}^2, \quad f(x) \in W_2^1(\mathbb{R}^2).$$

Then from the equality $\overline{H_0^{(1)}(\sqrt{\lambda}|x|)} = H_0^{(2)}(\sqrt{\lambda}|x|)$ [36] it follows that

$$\gamma(\lambda)\vec{c} = \frac{\pi i}{2} \sum_{k=1}^2 H_0^{(1)}(\sqrt{\lambda}|x - y_k|)c_k, \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{C}^2,$$

$$\gamma^*(\bar{\lambda})h(x) = -\frac{\pi i}{2} \left[\int_{\mathbb{R}^2} h(x) H_0^{(2)}(\sqrt{\lambda}|x - y_1|) dx \right. \\ \left. \int_{\mathbb{R}^2} h(x) H_0^{(2)}(\sqrt{\lambda}|x - y_2|) dx \right].$$

Set $r = |y_1 - y_2|$,

$$H(\lambda, r) = H_0^{(1)}(\sqrt{\lambda}r) - H_0^{(1)}(e^{3\pi i/4}r).$$

From (2.5), using unitarity of the Fourier transform, one can derive that the matrix $\mathcal{Q}(\lambda)$ in the standard basis is of the form:

$$\mathcal{Q}(\lambda) = \frac{\lambda - i}{\lambda + i} \pi \begin{bmatrix} -\ln(\lambda i) & \pi i H(\lambda, r) \\ \pi i H(\lambda, r) & -\ln(\lambda i) \end{bmatrix}.$$

Hence,

$$\mathcal{Q}^*(\lambda) = \frac{\bar{\lambda} + i}{\bar{\lambda} - i} \pi \begin{bmatrix} -\ln\left(\frac{\bar{\lambda}}{i}\right) & -\pi i \bar{H}(\lambda, r) \\ -\pi i \bar{H}(\lambda, r) & -\ln\left(\frac{\bar{\lambda}}{i}\right) \end{bmatrix}.$$

Now we will find the subspace \mathcal{D}_0 and the sesquilinear form $\Omega_0[\cdot, \cdot]$ (see Theorem 3.3).

$$\begin{aligned} (\mathcal{Q}(\lambda)\vec{c}, \vec{d}) &= \frac{\lambda - i}{\lambda + i} \pi \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}^* \begin{bmatrix} -\ln(\lambda i) & \pi i H(\lambda, r) \\ \pi i H(\lambda, r) & -\ln(\lambda i) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \frac{\lambda - i}{\lambda + i} \pi \left(-(c_1 \bar{d}_1 + c_2 \bar{d}_2) \ln(\lambda i) \right. \\ &\quad \left. + (c_2 \bar{d}_1 + c_1 \bar{d}_2) \pi i \left(H_0^{(1)}(\sqrt{\lambda}r) - H_0^{(1)}(e^{3\pi i/4}r) \right) \right). \end{aligned}$$

Taking into account the asymptotic behavior [36]

$$H_0^1(\lambda) = 1 + \frac{2i}{\pi} \left(\ln\left(\frac{\lambda}{2}\right) + \gamma \right) + o(\lambda), \quad \lambda \rightarrow 0,$$

where γ is Euler's constant, we see that

$$\mathcal{D}_0 := \left\{ e \in \mathcal{H} : \lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C} \setminus [0, +\infty)}} |(\mathcal{Q}(z)e, e)_{\mathcal{H}}| < \infty \right\} = \left\{ \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} \in \mathbb{C}^2 : \zeta \in \mathbb{C} \right\},$$

Let

$$\vec{c}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then

$$\begin{aligned}
\Omega_0[\zeta \vec{c}_0, \eta \vec{c}_0] &= -\zeta \bar{\eta} \lim_{\substack{\lambda \rightarrow 0 \\ \lambda < 0}} (\mathcal{Q}(\lambda) \vec{c}_0, \vec{c}_0) \\
&= \pi \zeta \bar{\eta} \lim_{\substack{\lambda \rightarrow 0 \\ \lambda < 0}} \left(-2 \ln(\lambda i) - 2\pi i (H_0^{(1)}(\sqrt{\lambda} r) - H_0^{(1)}(e^{3\pi i/4} r)) \right) \\
&= 2\pi \zeta \bar{\eta} \lim_{\substack{\lambda \rightarrow 0 \\ \lambda < 0}} \left(-\ln(\lambda i) - \pi i \left(1 + \frac{2i}{\pi} \left(\ln \left(\frac{\sqrt{\lambda} r}{2} \right) + \gamma \right) - H_0^{(1)}(e^{3\pi i/4} r) \right) \right) \\
&= 2\pi \zeta \bar{\eta} \lim_{\substack{\lambda \rightarrow 0 \\ \lambda < 0}} \left(-\ln(\lambda i) - \pi i + 2 \ln \left(\frac{\sqrt{\lambda} r}{2} \right) + 2\gamma + \pi i H_0^{(1)}(e^{3\pi i/4} r) \right) \\
&= 4\pi \zeta \bar{\eta} \left(\ln \frac{r}{2} - \frac{3\pi i}{4} + \gamma + \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4} r) \right) = \omega_0 \cdot \zeta \bar{\eta},
\end{aligned}$$

where

$$\omega_0 = 4\pi \left(\ln \frac{r}{2} - \frac{3\pi i}{4} + \gamma + \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4} r) \right).$$

From the latter equality one can obtain that

$$\operatorname{Re} \Omega_0[\zeta \vec{c}_0, \eta \vec{c}_0] = \operatorname{Re} \omega_0 \cdot \zeta \bar{\eta} = 4\pi \left(\ln \frac{r}{2} + \gamma + \mathbf{ker}(r) \right) \zeta \bar{\eta},$$

where the functions $\mathbf{ker}(\cdot)$ and $\mathbf{kei}(\cdot)$ are Kelvin functions [36, p.268], i.e., the real and imaginary parts of the function $\frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4}(\cdot))$, respectively:

$$\mathbf{ker}(r) + i \mathbf{kei}(r) = \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4} r).$$

For the operator-functions $\Phi(\lambda)$, $\mathcal{G}(\lambda)$, $\mathcal{Q}^*(\bar{\lambda})$, and $q(\lambda)$ on $\mathcal{D}_0 = \operatorname{dom}(\Omega_0)$ we have:

$$\begin{aligned}
\Phi(\lambda) X \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} &= \frac{\zeta}{4\pi^2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|p|}{|p|^2 - \lambda} \exp(i(x-y)p) g(y) dy dp, \\
\mathcal{G}(\lambda) X \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} &= -\frac{\pi i(\lambda + i)\zeta}{2} \left[\int_{\mathbb{R}^2} \Phi(\lambda)(f(x)) H_0^{(2)}(e^{3\pi i/4}|x - y_1|) dx \right. \\
&\quad \left. \int_{\mathbb{R}^2} \Phi(\lambda)(f(x)) H_0^{(2)}(e^{3\pi i/4}|x - y_2|) dx \right], \\
\mathcal{Q}^*(\bar{\lambda}) \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} &= \frac{\lambda + i}{\lambda - i} \pi \left(-\ln \left(\frac{\lambda}{i} \right) + \pi i \overline{H(\bar{\lambda}, |y_1 - y_2|)} \right) \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix}, \\
q(\lambda) \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} &= \frac{\pi i}{2} \frac{1}{i - \lambda} \zeta \left((i + \lambda)(H_0^{(1)}(\sqrt{\lambda}|x - y_2|) - H_0^{(1)}(\sqrt{\lambda}|x - y_1|)) \right. \\
&\quad \left. + 2i(H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|)) \right).
\end{aligned}$$

Now we find the operator $X_0 e = i \hat{A}_F^{-1/2} \gamma(i) e$, $e \in \mathcal{D}_0$ from Theorem 3.3. As was mentioned above it is convenient to use the momentum representation. Let $\hat{\gamma}(\lambda) =$

$\mathcal{F}\gamma(\lambda)$, where

$$\hat{f}(p) = (\mathcal{F}f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x) e^{-ix \cdot p} dx, \quad p = (p_1, p_2).$$

is the Fourier transform of $f(x) \in L_2(\mathbb{R}^2, dx)$. Then

$$\hat{\gamma}(\lambda) \vec{c} = \sum_{k=1}^2 c_k \frac{e^{-ipy_k}}{|p|^2 - \lambda}, \quad \forall \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{C}^2.$$

Hence,

$$\hat{X}_0 \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \mathcal{F}X_0 \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = i\hat{A}_F^{-1/2} \gamma(i) \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \frac{i(e^{-ipy_1} - e^{-ipy_2})}{|p|(|p|^2 - i)} \zeta,$$

So, $\hat{X}_0 \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \hat{g}_0(p) \zeta$, where

$$\hat{g}_0(p) = \frac{i(e^{-ipy_1} - e^{-ipy_2})}{|p|(|p|^2 - i)}. \quad (5.2)$$

Getting back to the coordinate representation, we obtain, using [40], [26, p.671], that

$$\begin{aligned} g_0(x) &= \mathcal{F}^{-1} \hat{g}_0(p) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{i(e^{ip(x-y_1)} - e^{ip(x-y_2)})}{|p|(|p|^2 - i)} dp = \\ &= i \int_0^{+\infty} \frac{J_0(\rho|x-y_1|) - J_0(\rho|x-y_2|)}{\rho^2 - i} d\rho = \\ &= \frac{\pi i}{2\sqrt{-i}} \left(I_0(\sqrt{-i}|x-y_1|) - L_0(\sqrt{-i}|x-y_1|) \right) \\ &\quad - \frac{\pi i}{2\sqrt{-i}} \left(I_0(\sqrt{-i}|x-y_2|) - L_0(\sqrt{-i}|x-y_2|) \right) = \\ &= -\frac{\pi}{2} e^{3\pi i/4} \left(\mathbf{M}_0(e^{-\pi i/4}|x-y_1|) - \mathbf{M}_0(e^{-\pi i/4}|x-y_2|) \right), \end{aligned}$$

where $I_0(\cdot)$ is the Bessel function and $L_0(\cdot), \mathbf{M}_0(\cdot)$ are modified Struve functions [36, p.288]. So,

$$X_0 \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = g_0(x) \zeta,$$

where

$$g_0(x) = -\frac{\pi}{2} e^{3\pi i/4} \left(\mathbf{M}_0(e^{-\pi i/4}|x-y_1|) - \mathbf{M}_0(e^{-\pi i/4}|x-y_2|) \right).$$

According to (3.4) we have

$$\|g_0(x)\|_{L_2(\mathbb{R}^2)}^2 = \operatorname{Re} \omega_0 = 4\pi \left(\ln \frac{r}{2} + \gamma + \mathbf{ker}(r) \right).$$

Remark 5.1. Since $\|g_0(x)\|_{L_2(\mathbb{R}^2)}^2 = \|\hat{g}_0(p)\|_{L_2(\mathbb{R}^2)}^2$ (the unitarity of the Fourier transform), expression (5.2) for $\hat{g}_0(p)$ gives

$$\|\hat{g}_0(p)\|_{L_2(\mathbb{R}^2)}^2 = 4\pi \int_0^\infty \frac{1 - J_0(r\rho)}{\rho(\rho^4 + 1)} d\rho.$$

On the other hand, due to (3.4), we have

$$\|g_0(x)\|_{L_2(\mathbb{R}^2)}^2 = \operatorname{Re} \omega_0.$$

This leads to the value of the improper integral $\int_0^\infty \frac{1 - J_0(r\rho)}{\rho(\rho^4 + 1)} d\rho$:

$$\int_0^\infty \frac{1 - J_0(r\rho)}{\rho(\rho^4 + 1)} d\rho = \frac{1}{4\pi} \operatorname{Re} \omega_0 = \left(\ln \frac{r}{2} + \gamma + \mathbf{ker}(r) \right).$$

In order to describe all m -sectorial extensions of A we need to define pairs $\langle \mathbf{Z}, X \rangle$ satisfying conditions 3), 4) from Theorem 2.4 and conditions 1)–3) of Theorem 3.8. Since \mathbf{Z} is m -accretive linear relation in \mathbb{C}^2 and $\operatorname{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0$, there are only two possible cases:

- 1) $\mathbf{Z} = \left\langle \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix}, z \cdot \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} \right\rangle \oplus \left\langle 0, \begin{bmatrix} \eta \\ \eta \end{bmatrix} \right\rangle$, $\zeta, \eta, z \in \mathbb{C}$, $\operatorname{Re} z \geq 0$;
- 2) $\mathbf{Z} = \langle 0, \mathbb{C}^2 \rangle$. As has been mentioned in [15] this linear relation corresponds to the Friedrichs extension A_F of A .

In the first case the operator X , acting from $\operatorname{dom}(\mathbf{Z})$ into $L_2(\mathbb{R}^2)$, takes the form $X \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \zeta g(x)$, where a function $g(x) \in L_2(\mathbb{R}^2)$ satisfies the condition

$$\|g(x)\|_{L_2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |g(x)|^2 dx \leq 2 \operatorname{Re} z. \quad (5.3)$$

For the form $\omega[\cdot, \cdot]$ defined by (3.10) we have

$$\begin{aligned} \omega[\zeta \vec{c}_0, \eta \vec{c}_0] &= (\mathbf{Z} \zeta \vec{c}_0, \eta \vec{c}_0) - \Omega_0[\zeta \vec{c}_0, \eta \vec{c}_0] - 2((X - X_0)\zeta \vec{c}_0, X_0 \eta \vec{c}_0) \\ &= \left(2z - \omega_0 - 2 \int_{\mathbb{R}^2} (g(x) - g_0(x)) \overline{g_0(x)} dx \right) \zeta \bar{\eta}. \end{aligned} \quad (5.4)$$

$$\operatorname{Re} \omega[\zeta \vec{c}_0] = \left(2 \operatorname{Re} z + \int_{\mathbb{R}^2} |g(x) - g_0(x)|^2 dx - \int_{\mathbb{R}^2} |g(x)|^2 dx \right) |\zeta|^2.$$

Thus, the form $\omega[\cdot, \cdot]$ is determined by the number

$$w_{\langle z, g(x) \rangle} = 2z - \omega_0 - 2 \int_{\mathbb{R}^2} (g(x) - g_0(x)) \overline{g_0(x)} dx. \quad (5.5)$$

Clearly, the form $\omega[\cdot, \cdot]$ is sectorial iff

$$\begin{aligned} \operatorname{Re} w_{\langle z, g(x) \rangle} &= 2 \operatorname{Re} z + \int_{\mathbb{R}^2} |g(x) - g_0(x)|^2 dx - \int_{\mathbb{R}^2} |g(x)|^2 dx > 0 \\ \text{or } w_{\langle z, g(x) \rangle} &= 0. \end{aligned} \quad (5.6)$$

Remark 5.2. Due to $2 \operatorname{Re} z - \int_{\mathbb{R}^2} |g(x)|^2 dx \geq 0$ the equality $w_{\langle z, g(x) \rangle} = 0$ implies that $g(x) = g_0(x)$ almost everywhere and $z = \omega_0/2$.

Further, condition 3) from Theorem 5 takes the form

$$M \int_{\mathbb{R}^2} |g(x) - g_0(x)|^2 dx \leq 2 \operatorname{Re} z - \int_{\mathbb{R}^2} |g(x)|^2 dx,$$

where $M > 0$. The latter inequality can be simplified as follows

$$2\operatorname{Re} z - \int_{\mathbb{R}^2} |g(x)|^2 dx > 0. \quad (5.7)$$

So, conditions (5.3), (5.6) are satisfied. Note, that in this case linear relation $\mathbf{W}(\lambda)$, see (2.11), is the of the form

$$\begin{aligned} \mathbf{W}(\lambda) = & \left\langle \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix}, \left(z - \frac{\lambda + i}{\lambda - i} \pi \left(-\ln \left(\frac{\lambda}{i} \right) + \pi i \overline{H(\bar{\lambda}, |y_1 - y_2|)} \right) \right) \cdot \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} \right. \\ & \left. - \pi i(\lambda + i) \zeta \left[\int_{\mathbb{R}^2} \Phi(\lambda)(g(x)) H_0^{(2)}(e^{3\pi i/4} |x - y_1|) dx \right] \right. \\ & \left. + \int_{\mathbb{R}^2} \Phi(\lambda)(g(x)) H_0^{(2)}(e^{3\pi i/4} |x - y_2|) dx \right] + \begin{bmatrix} \eta \\ \eta \end{bmatrix} \right\rangle. \end{aligned}$$

for all $\lambda \in \rho(A_F) = \mathbb{C} \setminus [0, +\infty)$. Then

$$\mathbf{W}^{-1}(\lambda) = \left\langle \begin{bmatrix} \zeta \\ \eta \end{bmatrix}, \frac{1}{w_{\langle z, g(x) \rangle}(\lambda)} \begin{bmatrix} \zeta - \eta \\ -\zeta + \eta \end{bmatrix} \right\rangle,$$

where

$$\begin{aligned} w_{\langle z, g(x) \rangle}(\lambda) = & 2 \left(z - \frac{\lambda + i}{\lambda - i} \pi \left(-\ln \left(\frac{\lambda}{i} \right) + \pi i \overline{H(\bar{\lambda}, |y_1 - y_2|)} \right) \right) \\ & - \pi i(\lambda + i) \int_{\mathbb{R}^2} \Phi(\lambda)(g(x)) \left(H_0^{(2)}(e^{3\pi i/4} |x - y_1|) - H_0^{(2)}(e^{3\pi i/4} |x - y_2|) \right) dx. \end{aligned}$$

Clearly, $\ker(\mathbf{W}(\lambda)) \neq \{0\}$ iff $w_{\langle z, g(x) \rangle}(\lambda) = 0$ and

$$\ker(\mathbf{W}(\lambda)) = \operatorname{dom}(\mathbf{W}(\lambda)) = \begin{bmatrix} \eta \\ -\eta \end{bmatrix}, \quad \eta \in \mathbb{C}.$$

Let an m -sectorial extension \tilde{A} of A be defined by a pair $\langle z, g(x) \rangle$, satisfying (5.7), see Theorem (3.8). Since \tilde{A} is m -sectorial extension and

$$G(-i) = 0, \quad \mathcal{Q}^*(-i) = 0, \quad q(-i) = \gamma(i),$$

it is suitable to take $\lambda = -i$ and apply Theorem 2.5, Remark 2.6, and equalities (2.13), (2.14), (2.15). Then,

$$\begin{aligned} \mathbf{W} = \mathbf{W}(-i) = \mathbf{Z} = & \left\langle \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix}, z \cdot \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} \right\rangle \oplus \left\langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \eta \\ \eta \end{bmatrix} \right\rangle, \\ \mathbf{W}^{-1} = & \left\langle \begin{bmatrix} \zeta \\ \eta \end{bmatrix}, \frac{1}{2z} \begin{bmatrix} \zeta - \eta \\ -\zeta + \eta \end{bmatrix} \right\rangle. \end{aligned}$$

By (2.13)

$$\operatorname{dom}(\tilde{A}) = \left(I + (q(-i) - 2\Phi(-i)X) \mathbf{W}^{-1}(-i) \gamma^*(-i) (A_F + iI) \right) \operatorname{dom}(A_F).$$

Further, let $\delta(x)$, $x = (x_1, x_2)$ be the Dirac delta. Then $\delta(x) \in W_2^{-2}(\mathbb{R}^2)$ [2]. Since $\mathcal{F}(\delta(x)) = 1/2\pi$, then $\mathcal{F}^{-1}(1) = 2\pi\delta(x)$. So, if $\mathcal{F}(h(x)) = \hat{h}(p)$ and $h(x) \in \operatorname{dom}(A_F) = W_2^2(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} (e^{ipy_1} - e^{ipy_2}) \hat{h}(p) dp = 2\pi(h(y_1) - h(y_2)).$$

Using the latter equality and the Fourier transform we obtain that

$$\mathbf{W}^{-1}(-i)\gamma^*(-i)(A_F + iI)h(x) = \frac{\pi(h(y_1) - h(y_2))}{z}.$$

If $h(x) \in \text{dom}(A_F)$, then

$$\text{dom}(\tilde{A}) = \left\{ \begin{aligned} &h(x) + \frac{\pi(h(y_1) - h(y_2))}{z} \\ &\times \left(\frac{\pi i}{2} \left(H_0^{(1)}(e^{3\pi i/4}|x - y_1|) - H_0^{(1)}(e^{3\pi i/4}|x - y_2|) \right) \right. \\ &\quad \left. - 2\Phi(-i)(g(x)) \right) \end{aligned} \right\}.$$

Then applying Theorems 2.5, 3.8 we arrive at the following statement.

Theorem 5.3. *There is a bijective correspondence between all m -sectorial extensions \tilde{A} (except Friedrichs and Kreĭn-von Neumann extensions) of A given by (5.1) and all pairs $\langle z, g(x) \rangle$, where $z \in \mathbb{C}$ and a function $g(x) \in L_2(\mathbb{R}^2)$ are such that:*

$$\|g(x)\|_{L_2(\mathbb{R}^2)}^2 < 2\text{Re } z.$$

This correspondence is given by the relations:

$$\text{dom}(\tilde{A}) = \left\{ \begin{aligned} &u(x) = h(x) + \frac{\pi(h(y_1) - h(y_2))}{z} \times \\ &\quad \times \left(\frac{\pi i}{2} \left(H_0^{(1)}(e^{3\pi i/4}|x - y_1|) - H_0^{(1)}(e^{3\pi i/4}|x - y_2|) \right) \right. \\ &\quad \left. - 2\Phi(-i)(g(x)) \right), \\ &h(x) \in W_2^2(\mathbb{R}^2) \end{aligned} \right\},$$

$$\begin{aligned} \tilde{A}u(x) &= -\Delta h(x) - i \frac{\pi(h(y_1) - h(y_2))}{z} \\ &\quad \times \left(\frac{\pi i}{2} \left(H_0^{(1)}(e^{3\pi i/4}|x - y_1|) - H_0^{(1)}(e^{3\pi i/4}|x - y_2|) \right) - 2\Phi(-i)(g(x)) \right). \end{aligned}$$

Moreover,

- 1) a number $\lambda \in \mathbb{C} \setminus [0, +\infty)$ is a regular point of \tilde{A} if and only if $w_{\langle z, g(x) \rangle}(\lambda) \neq 0$ and,

$$\begin{aligned} (\tilde{A} - \lambda I)^{-1}h(x) &= \frac{i}{4} \int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x - y|)f(y)dy + \frac{1}{w_{\langle z, g(x) \rangle}} \\ &\quad \times \left(\frac{\pi i}{2} \frac{1}{i - \lambda} \left((i + \lambda)(H_0^{(1)}(\sqrt{\lambda}|x - y_2|) - H_0^{(1)}(\sqrt{\lambda}|x - y_1|)) \right. \right. \\ &\quad \left. \left. + 2i (H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|)) \right) - 2\Phi(\lambda)(g(x)) \right) \end{aligned}$$

$$\times \left(-\frac{\pi i}{2}\right) \int_{\mathbb{R}^2} \left(H_0^{(2)}(\sqrt{\lambda}|x - y_1|) - H_0^{(2)}(\sqrt{\lambda}|x - y_2|)\right) h(x) dx.$$

2) a number $\lambda \in \rho(A_F)$ is an eigenvalue of \tilde{A} if and only if $w_{\langle z, g(x) \rangle}(\lambda) = 0$ and,

$$\begin{aligned} \ker(\tilde{A} - \lambda I) = & \left(\frac{\pi i}{2} \frac{1}{i - \lambda} \left((i + \lambda)(H_0^{(1)}(\sqrt{\lambda}|x - y_2|) - H_0^{(1)}(\sqrt{\lambda}|x - y_1|)) \right. \right. \\ & \left. \left. + 2i (H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|)) \right) - 2\Phi(\lambda)(g(x)) \right) \eta, \quad \eta \in \mathbb{C}. \end{aligned}$$

Corollary 5.4. *Let A be given by (5.1). Then there is a bijective correspondence between all m -accretive quasi-selfadjoint extensions \tilde{A} of A (except Friedrichs and Kreĭn-von Neumann extensions) and all complex numbers $z \in \mathbb{C}$ such that:*

$$\operatorname{Re} z \geq 2\pi \left(\ln \frac{|y_1 - y_2|}{2} + \gamma + \mathbf{ker}(|y_1 - y_2|) \right).$$

Moreover, an extension \tilde{A} is m -sectorial if and only if

$$\operatorname{Re} z > 2\pi \left(\ln \frac{|y_1 - y_2|}{2} + \gamma + \mathbf{ker}(|y_1 - y_2|) \right),$$

and is nonnegative selfadjoint if and only if

$$\operatorname{Im} z = \pi (-3\pi + 4 \mathbf{kei}(|y_1 - y_2|))$$

The correspondence is given by relations

$$\operatorname{dom}(\tilde{A}) = \left\{ \begin{aligned} & u(x) = h(x) + \frac{\pi(h(y_1) - h(y_2))}{z} \times \\ & \times \left(\frac{\pi i}{2} \left(H_0^{(1)}(e^{3\pi i/4}|x - y_1|) - H_0^{(1)}(e^{3\pi i/4}|x - y_2|) \right) \right), \\ & h(x) \in W_2^2(\mathbb{R}^2) \end{aligned} \right\}, \quad (5.8)$$

$$\begin{aligned} \tilde{A}u(x) = & -\Delta h(x) + \frac{\pi^2(h(y_1) - h(y_2))}{2z} \times \\ & \times \left(H_0^{(1)}(e^{3\pi i/4}|x - y_1|) - H_0^{(1)}(e^{3\pi i/4}|x - y_2|) \right). \end{aligned} \quad (5.9)$$

Moreover,

1) a number $\lambda \in \mathbb{C} \setminus [0, +\infty)$ is a regular point of \tilde{A} if and only if

$$w(z, \lambda) = z - \pi \ln(\lambda i) - \pi^2 i (H_0^{(1)}(\sqrt{\lambda}|y_1 - y_2|) - H_0^{(1)}(e^{3\pi i/4}|y_1 - y_2|)) \neq 0$$

and,

$$\begin{aligned} (\tilde{A} - \lambda I)^{-1}h(x) = & \frac{i}{4} \int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x - y|) f(y) dy \\ & + \frac{\pi^2}{8w(z, \lambda)} \times \left(H_0^{(1)}(\sqrt{\lambda}|x - y_1|) - H_0^{(1)}(\sqrt{\lambda}|x - y_2|) \right) \\ & \times \int_{\mathbb{R}^2} \left(H_0^{(2)}(\sqrt{\lambda}|x - y_1|) - H_0^{(2)}(\sqrt{\lambda}|x - y_2|) \right) h(x) dx. \end{aligned}$$

2) a number $\lambda \in \mathbb{C} \setminus [0, +\infty)$ is an eigenvalue of \tilde{A} if and only if $w(z, \lambda) = 0$ and,

$$\ker(\tilde{A} - \lambda I) = \left(H_0^{(1)}(\sqrt{\lambda}|x - y_1|) - H_0^{(1)}(\sqrt{\lambda}|x - y_2|) \right) \eta, \quad \eta \in \mathbb{C}.$$

Remark 5.5. One can obtain a description of the Kreĭn-von Neumann extension A_N of A from relations (5.8), (5.9) by substituting

$$2z = \omega_0 = 4\pi \left(\ln \frac{|y_1 - y_2|}{2} - \frac{3\pi i}{4} + \gamma + \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4}|y_1 - y_2|) \right).$$

It follows from (4.1) that form $A_N[u, v]$ associated with the Kreĭn-von Neumann extension A_N takes the form

$$D[A_N] = \left\{ \begin{array}{l} u(x) = h(x) + \frac{\pi i}{2} \left(H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \omega, \\ h(x) \in W_2^1(\mathbb{R}^2), \quad \omega \in \mathbb{C} \end{array} \right\},$$

and if

$$\begin{aligned} u(x) &= h_1(x) + \frac{\pi i}{2} \left(H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \omega_1, \\ v(x) &= h_2(x) + \frac{\pi i}{2} \left(H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \omega_2, \end{aligned}$$

where $h_1(x), h_2(x) \in W_2^1(\mathbb{R}^2)$, $\omega_1, \omega_2 \in \mathbb{C}$, then

$$\begin{aligned} A_N[u, v] &= \int_{\mathbb{R}^2} \nabla h_1(x) \overline{\nabla h_2(x)} dx \\ &\quad - \frac{\pi \bar{\omega}_2}{2} \int_{\mathbb{R}^2} h_1(x) \overline{\left(H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right)} dx \\ &\quad - \frac{\pi \omega_1}{2} \int_{\mathbb{R}^2} \left(H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \overline{h_2(x)} dx \\ &\quad + 4\pi \left(\ln \frac{|y_1 - y_2|}{2} + \gamma + \mathbf{ker} |y_1 - y_2| \right) \cdot \omega_1 \bar{\omega}_2. \end{aligned}$$

REFERENCES

- [1] V. Adamyan, *Nonnegative Perturbations of Nonnegative Self-adjoint Operators*, Methods Funct. Anal. Topology. **13** (2007), No.2, 103-109.
- [2] S. Albeverio, F. Gestezy, R. Hoegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Springer, New York, 1988.
- [3] T. Ando and K. Nishio, *Positive selfadjoint extensions of positive symmetric operators*, Tohoku Math. J. **22** (1970), 65-75.
- [4] R. Arens, *Operational calculus of linear relations*. Pacific J. Math., **11** (1961), 9-23.
- [5] Yu. M. Arlinskiĭ, *Positive spaces of boundary values and sectorial extensions of nonnegative symmetric operators*, Ukrain. Math. Zh, **40** (1988), no.1, 8-14 (Russian). English translation in Ukr. Math. J. **40** (1988), No.1, 5-10.
- [6] Yu. M. Arlinskiĭ, *On proper accretive extensions of positive linear relations*, Ukr. Math. Journ. **47** (1995), No.6, 723-730.
- [7] Yu. M. Arlinskiĭ, *Maximal sectorial extensions and closed forms associated with them*, Ukrainian Math. J. **48** (1996), 809-827.

- [8] Yu. M. Arlinskiĭ, *Extremal extensions of sectorial linear relations*, Matematichnii Studii **7** (1997), 81–96.
- [9] Yu. M. Arlinskiĭ, *On m -accretive extensions and restrictions*, Methods of Funct. Anal. Topol. **4** (1998), 1–26.
- [10] Yu. M. Arlinskiĭ, *On functions connected with sectorial operators and their extensions*, Int. Equat. Oper. Theory **33** (1999), 125–152.
- [11] Yu. M. Arlinskiĭ, *Abstract boundary conditions for maximal sectorial extensions of sectorial operators*, Math. Nachr. **209** (2000), 5–36.
- [12] Yu. Arlinskiĭ, *Extremal extensions of a $C(\alpha)$ -suboperator and their representations*, Oper. Theory Adv. Appl., **162** (2006), 47–69.
- [13] Yu. Arlinskiĭ, *Boundary triplets and maximal accretive extensions of sectorial operators*, in “Operator Methods for Boundary Value Problems”, London Mathematical Society Lecture Note Series, No. 404, 2012, 33–70. Cambridge University Press.
- [14] Yu. Arlinskiĭ, Yu. Kovalev, and E. Tsekanovskii, *Accretive and sectorial extensions of nonnegative symmetric operators*, CAOT, **6** (2012), No.3, 677–718.
- [15] Yu. M. Arlinskiĭ and A.B. Popov, *On m -accretive extensions of sectorial operator*, Math. Sb. **204** (2013), No.8, 3–40 (Russian). English translation in Sbornik: Mathematics, **204:8** (2013), 1085–1121.
- [16] Yu. M. Arlinskiĭ and E. R. Tsekanovskii, *Quasi-selfadjoint contractive extensions of Hermitian contractions*, Teor. Funkts., Funkts. Anal. Prilozhen, **50** (1988), 9–16 (Russian). English translation in J. Math. Sci. **49** (1990), No.6, 1241–1247.
- [17] Yu. Arlinskiĭ and E. Tsekanovskii, *The von Neumann problem for nonnegative symmetric operators*, Int. Eq. and Oper. Theory, **51** (2005), 319–356.
- [18] Yu. Arlinskiĭ and E. Tsekanovskii, *Krein’s research on semi-bounded operators, its contemporary developments, and applications*. Operator Theory: Advances and Applications, **190** (2009), 65–112.
- [19] Ashbaugh, M., Gesztesy, F., Mitrea, M., Shterenberg, R., Teschl, G.: *A Survey on the Kreĭn-von Neumann Extension, the Corresponding Abstract Buckling Problem, and Weyl-type Spectral Asymptotics for Perturbed Kreĭn Laplacians in Nonsmooth Domains*, Operator Theory: Advances and Applications, **232** (2013), 1–106.
- [20] E. A. Coddington and H. S. V. de Snoo, *Positive selfadjoint extensions of positive symmetric subspaces*, Math. Z. **159** (1978), 203–214.
- [21] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal., **95** (1991), No.1, 1–95.
- [22] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. of Math. Sci., **73** (1995), No.2, 141–242.
- [23] V. A. Derkach, M. M. Malamud, and E. R. Tsekanovskii, *Sectorial extensions of positive operators and characteristic functions*, Ukrainian Math. Journ. **41** (1989), no.2, 151–158 (Russian). English translation in Ukr. Math. J. **41** (1989), No.2, 136–142.
- [24] F. Gesztesy, N. Kalton, K. Makarov, E. Tsekanovskii, *Some applications of operator-valued Herglotz functions*, Oper. Theory, Adv. and Appl., **123** (2001), 271–321.
- [25] M. L. Gorbachuk and V. I. Gorbachuk, *Boundary value problems for differential-operator equations*, Naukova Dumka, Kiev, 1984 (Russian). English translation: Kluwer Academic Publishers, 1991.
- [26] A. Jeffrey and D. Zwillinger, *Table of integrals, series, and products*, Table of Integrals, Series, and Products Series, Elsevier Science, 2007.
- [27] T. Kato, *Fractional powers of dissipative operators*, Journ. Math. Soc. Japan **13** (1961), 248–274.
- [28] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin–Heidelberg–New York, 1966.
- [29] A. N. Kochubei, *Extensions of a positive definite symmetric operator*. Dokl. Akad. Nauk Ukrain. SSR, Ser. A, no. 3, 1979, 168–171 (Russian)

- [30] M. G. Kreĭn, *The theory of selfadjoint extensions of semibounded hermitian transformations and its applications. I*, Mat.Sbornik **20** (1947), 431–495, (Russian).
- [31] M. G. Kreĭn, *The theory of selfadjoint extensions of semibounded hermitian transformations and its applications. II*, Mat.Sbornik **21** (1947), 365–404, (Russian).
- [32] M.G.Kreĭn and H.Langer, *Über die Q - function eines Π - Hermiteschen operators im raum Π_κ* , Acta Sci. Math. Szeged **34** (1973), 191-230.
- [33] M.G.Kreĭn and H.Langer, *On defect subspaces and generalized resolvents of Hermitian operator in the space Π_κ* , Fuctional Analysis and Appl. **5**, No.3 (1971), No.3, 54-69 (Russian).
- [34] M.M.Malamud, *On some classes of Hermitian operators with gaps*, Ukrainian Mat.Zhu., **44** (1992), No.2, 215–234 (Russian). English translation: *Ukr. Math. J.*, **44** (1992), 1522–1547.
- [35] V.A. Mikhalets, *Solvable and sectorial boundary value problems for the operator Sturm-Liouville equation*. Ukrainian Math Zh., **26** (1974), 450-459 (Russian).
- [36] F.W.J. Olver, National Institute of Standards, and Technology (U.S.), *Nist handbook of mathematical functions*, Cambridge University Press, 2010.
- [37] R. Phillips, *Dissipative operators and hyperbolic system of partial differential equations*, Trans. Amer. Math. Soc. **90** (1959), 192–254.
- [38] R.S. Phillips, *On dissipative operators*, Lectures in differential equations **3** (1969), 65–113.
- [39] F.S. Roĭe-Beketov, *Numerical range of a linear relation and maximal relations*, Theory of Functions, Functional Anal. and Appl., **44**, (1985), 103–112 (Russian). English translation in J. Soviet Math., **48** (1990), No.3, 329–336.
- [40] I.N. Sneddon, *Fourier transforms*, Dover Books on Mathematics, Dover Publications, 1995.
- [41] E.R. Tsekanovskii, *Non-selfadjoint accretive extenions of positive operators and theorems of Friedrichs–Kreĭn–Phillips*, Functional Anal. i Prilozhen. **14** (1980), No.2, 87–88 (Russian). English translation in Funct. Anal. Appl. **14:2** (1980), 156–157.

DEPARTMENT OF MATHEMATICAL ANALYSIS, EAST UKRAINIAN NATIONAL DAHL UNIVERSITY,
PROSPECT SOVETSKII, 59-A, SEVERODONETSK, 93400, UKRAINE

E-mail address: yury.arlinskii@gmail.com

DEPARTMENT OF MATHEMATICAL ANALYSIS, EAST UKRAINIAN NATIONAL DAHL UNIVERSITY,
PROSPECT SOVETSKII, 59-A, SEVERODONETSK, 93400, UKRAINE

E-mail address: Andrey.B.Popov@gmail.com